# The McGucken Principle $dx_4/dt = ic$ Necessitates the Wick Rotation and $i$ Throughout Physics:

A Reduction of Thirty-Four Independent Inputs of Quantum Field Theory, Quantum Mechanics, and Symmetry Physics to a Single Physical Principle, with the Imaginary Unit $i$ Identified Across All of Physics — Including Penrose's Twistors, the Arkani-Hamed–Trnka Amplituhedron, Feynman Diagrams, AdS/CFT, String Theory, and the Extra Dimensions of Kaluza–Klein, String Theory, M-Theory, and AdS/CFT — as the Algebraic Signature of the Fourth Expanding Dimension

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*"More intellectual curiosity, versatility and yen for physics than Elliot McGucken's I have never seen in any senior or graduate student. Originality, powerful motivation, and a can-do spirit make me think that McGucken is a top bet..." — Dr. John Archibald Wheeler, Joseph Henry Professor of Physics, Princeton University*

*"Behind it all is surely an idea so simple, so beautiful, that when we grasp it — in a decade, a century, or a millennium — we will all say to each other, how could it have been otherwise?" — John Archibald Wheeler*

Eppur si muove. *(And yet it moves.) — Galileo Galilei, of the Earth. So too of the fourth dimension.*

*"The square root of minus one means that nature works with complex numbers and not with real numbers." — Freeman Dyson, "Birds and Frogs," 2008*

*Dyson named the joke. The McGucken Principle identifies it: every i in physics is the algebraic record of the fourth dimension expanding at the velocity of light.*

## Abstract

This paper establishes that the McGucken Principle, which states that the fourth dimension is expanding at the velocity of light in a spherically symmetric manner $dx_4/dt = ic$, necessitates the Wick rotation as a direct consequence of its geometric content. The imaginary unit $i$ throughout physics is the signature of $x_4$'s expansion perpendicular to the three spatial dimensions — it is the signature of $dx_4/dt = ic$; it exalts the McGucken Sphere [28, 41, 42, 45] as the foundational atom of spacetime and makes the McGucken Symmetries [27, 35] possible. Every instance in which the Wick rotation appears throughout theoretical physics — the Wick substitution $t \to -i\tau$ itself, the convergence of the Euclidean path integral, the $+i\varepsilon$ prescription, the Schrödinger-to-diffusion correspondence, Osterwalder–Schrader reflection positivity, the Kubo–Martin–Schwinger condition, Gibbons–Hawking horizon regularity, the Hawking temperature, the Matsubara formalism, and the twelve distinct factor-of-$i$ insertions across canonical quantization, the Schrödinger equation, the canonical commutator $[\hat{q}, \hat{p}] = i\hbar$, the Dirac equation, the path integral weight $e^{iS/\hbar}$, Fresnel integrals, the Minkowski–Euclidean action bridge $iS_M = -S_E$, U(1) gauge phase, spinor structure, the KMS condition, and the Born rule $P = |\psi|^2$ — descends from this single Principle as a theorem. All this further attests to the Truth of the physical invariant $dx_4/dt = ic$. Thirteen formal theorem-clusters comprising thirty-four individual propositions are proved. The reduction is not partial: the McGucken Principle does not justify the Wick rotation, it constitutes it. The Principle and the rotation are the same geometric fact expressed in two coordinate systems. We establish this first for flat spacetime and the path integral, then for static curved backgrounds and horizon thermodynamics, then for thermal field theory and the KMS condition, then for the canonical commutator and the Born rule of quantum mechanics. The Kontsevich–Segal 2021 holomorphic-semigroup characterization of admissible complex metrics, which required two independent inputs — the semigroup structure and a separate positivity axiom — is identified as the formal shadow of the McGucken real rotation family projected into complex-metric language, with the positivity axiom emerging as a consequence of $x_4$ being a real axis supporting a real action. One Principle replaces two. The paper extends the unification beyond the Wick-rotation domain to establish that the imaginary unit $i$ throughout physics — in symmetries and conservation laws (the McGucken Symmetry as Father Symmetry of physics [27, 35], beneath the Lorentz group, the Poincaré group, the gauge groups U(1)×SU(2)×SU(3), the Wigner mass-spin classification, CPT, supersymmetry, and the standard string-theoretic dualities); in the foundational atom of spacetime (the McGucken Sphere [28, 41, 42], simultaneously realizing Huygens' secondary wavefront, the forward light cone, the Penrose twistor space $\mathbb{CP}^3$, and the Arkani-Hamed–Trnka amplituhedron); in the Dirac equation (with spin-$\frac{1}{2}$, the SU(2) double cover, and matter-antimatter as theorems of $x_4$-rotation [29]); and in quantum mechanics itself (with the canonical commutator and the Born rule derived through dual structurally-disjoint channels [33, 43, 32, 30]) — is in every case the algebraic signature of the fourth expanding axis acting through whatever derivation chain produces the expression in which $i$ appears. The foundational status is determined by axiomatic economy combined with constitutive rather than deductive derivation. On the terrain of the Wick rotation and its applications, of symmetries and conservation laws, of the foundational atom of spacetime, of the Dirac equation, and of the canonical structure of quantum mechanics, the McGucken Principle is the primitive structure of which every prior account has been a formal description.

The imaginary unit $i$ in the McGucken Principle $dx_4/dt = ic$ encodes a foundational fact about the structure of the universe [27, 35, 56]: $dx_4/dt = ic$ is not only the universe's foundational invariant — the fourth expanding dimension at the velocity of light from which every other invariant of physics descends as a theorem [27, 36, 30] — but is simultaneously the universe's foundational *asymmetry*. The factor $i$ distinguishes $x_4$ from the three spatial dimensions $(x_1, x_2, x_3)$ in that $x_4$ alone has motion built into its very definition; the factor of $c$ specifies that this motion is at the velocity of light; and the directionality of the advance — $dx_4/dt = +ic$ rather than $-ic$ — shows that the universe is governed by $x_4$'s one-way expanse [28, 45, 39]. Every irreversibility in physics, every arrow of time, every distinction between the spatial and the temporal, every imaginary structure in physical equations, descends from this single asymmetry [39, 37]. Symmetry and asymmetry, invariance and directionality, the geometric and the algebraic [54], and the Seven McGucken Dualities [55], are unified in the single Principle $dx_4/dt = ic$, as are all motivations for the Wick rotation.

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# 1 The McGucken Principle

The fourth coordinate $x_4 = ict$ of Minkowski spacetime, written by Minkowski in his 1908 Cologne address [9] and present in every standard treatment of special relativity since, is read here at face value: as a real geometric axis, with the imaginary factor $i$ as the algebraic marker of perpendicularity to the three spatial dimensions, and the differential consequence $dx_4/dt = ic$ as a dynamical statement about the universe's geometric structure [56, 27, 35, 28, 46]. Minkowski is way downstream from the physical invariant $dx_4/dt = ic$: Minkowski's identity supplies the static algebraic notation alone, while $dx_4/dt = ic$ supplies the physical content from which the symmetries of physics [27, 35, 54], the foundational atom of spacetime [28, 45], the moving-dimension geometry of the underlying manifold [46], the McGucken Space and McGucken Operator $D_M$ [34], the Wick rotation [26], general relativity [36], quantum mechanics [30], thermodynamics [37], the Dirac equation [29], the Born rule [43], the canonical commutator [33, 32], the broken symmetries and arrows of time [39], the cosmological observations [40], and Minkowski's identity itself all descend as theorems [38]. To paraphrase first-man-on-the-moon Neil Armstrong's "One small step for man, one giant leap for mankind": differentiating Minkowski's $x_4 = ict$ is one small step for math, and seeing the vast physical significance of $dx_4/dt = ic$ is one giant leap for physics.

**Principle 1** (The McGucken Principle)**.** The fourth coordinate $x_4$ of Minkowski spacetime is a physical geometric axis advancing at the invariant rate

$$\frac{dx_4}{dt} = ic,$$

with the advance proceeding from every spacetime event simultaneously and spherically symmetrically. Equivalently, $x_4 = ict$, where the imaginary factor is the algebraic marker of $x_4$'s geometric perpendicularity to the three spatial dimensions $x_1, x_2, x_3$. The Principle is the foundational invariant from which every other invariant of physics descends as a theorem [56, 27, 35].

The Principle promotes Minkowski's 1908 notation $x_4 = ict$ [9] from a bookkeeping device that generates the Lorentzian signature to a physical statement about a fourth geometric axis [56, 27]. Under the Principle, spacetime is a four-dimensional Euclidean manifold $M$ [46, 34] with line element

$$d\ell^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2. \tag{1}$$

Substitution of $x_4 = ict$ into (1) gives

$$d\ell^2 = dx_1^2 + dx_2^2 + dx_3^2 + (ic)^2\, dt^2 = dx_1^2 + dx_2^2 + dx_3^2 - c^2dt^2 = ds^2, \tag{2}$$

the Lorentzian line element. The apparent Lorentzian signature of spacetime is therefore the signature that arises when the real four-dimensional Euclidean geometry is projected onto three spatial coordinates and a coordinate time $t$, with $x_4$ absorbed into $t$ via the Minkowski identity. The minus sign in front of $c^2dt^2$ is the algebraic image of the $i$ in $x_4 = ict$, squared. This is the first appearance of what will become the central claim of this paper: every minus sign and every factor of $i$ throughout the formalism of relativistic and quantum physics is a record of the same geometric fact.

**Theorem 1.1** (The $i$ in Minkowski spacetime is the integrated signature of $dx_4/dt = ic$)**.** *The factor of $i$ in Minkowski's 1908 identity $x_4 = ict$ [9], the minus sign in the Lorentzian metric signature (2), and the imaginary character of the proper time differential $d\tau = dx_4/(ic)$ are all algebraic consequences of the McGucken Principle $dx_4/dt = ic$. Specifically:*

*(a) Integration of $dx_4/dt = ic$ from $t = 0$ with $x_4(0) = 0$ gives $x_4 = ict$, recovering Minkowski's identity from the McGucken Principle as a direct first-order consequence.*

*(b) Substitution of $x_4 = ict$ into the real positive-definite Euclidean line element $d\ell^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$ yields $ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - c^2 dt^2$, the Lorentzian line element. The minus sign in the Lorentzian signature is $i^2 = -1$, the algebraic square of the imaginary factor in $dx_4/dt = ic$.*

*(c) The imaginary unit $i$ in Minkowski's identity is the algebraic perpendicularity-marker of $x_4$ relative to the three spatial dimensions: rotation by $i$ in the complex plane is rotation by $\pi/2$, and the McGucken Principle states that $x_4$ advances at the perpendicular angle $\pi/2$ (the imaginary direction) at speed $c$.*

*The $i$ that appears in every formula of relativistic and quantum physics — in the Lorentzian metric signature, in canonical quantization $\hat{E} = i\hbar\,\partial/\partial t$, in the Schrödinger equation, in the Dirac equation, in the path integral weight $e^{iS/\hbar}$, in the U(1) gauge phase, in the unitary evolution operator $e^{-i\hat{H}t/\hbar}$, in the canonical commutator $[\hat{q}, \hat{p}] = i\hbar$, in the worldsheet complex structure of string theory, in twistor space $\mathbb{CP}^3$, in the amplituhedron, and in every other imaginary structure throughout physics — traces through Minkowski's identity $x_4 = ict$ to the McGucken Principle $dx_4/dt = ic$ as its differential source. The McGucken Sphere [28, 41, 42] is the foundational atom of spacetime that this $i$ exalts; the McGucken Symmetries [27, 35] are the symmetries this $i$ makes possible.*

*Proof.* (a) Direct integration: $dx_4/dt = ic \implies x_4(t) - x_4(0) = ict$, with the convention $x_4(0) = 0$ giving Minkowski's identity. (b) Direct substitution of $x_4 = ict$ into the Euclidean line element, with $dx_4 = ic\,dt$ and $dx_4^2 = (ic)^2\,dt^2 = -c^2\,dt^2$, yields the Lorentzian form. (c) The geometric interpretation of multiplication by $i$ in the complex plane as a $\pi/2$ rotation is standard (Argand 1806, Wessel 1797); the identification of $x_4$ as the axis perpendicular to the three spatial dimensions, advancing at rate $c$, makes this geometric content physical. The terminal claim that every $i$ in relativistic and quantum physics traces through Minkowski's identity to the McGucken Principle is established constructively in §§2–14 of this paper, where each instance of $i$ is identified by name and its derivation chain to $dx_4/dt = ic$ is exhibited. □

**Remark 1.2.** Theorem 1.1 clarifies the logical relationship between Minkowski's 1908 identity $x_4 = ict$ and the McGucken Principle $dx_4/dt = ic$. Minkowski's identity is the integrated form of the McGucken Principle; the McGucken Principle is the differential form of Minkowski's identity. The two statements are mathematically equivalent under the boundary condition $x_4(0) = 0$. What Minkowski supplied in 1908 is the static geometric statement; what the McGucken Principle adds is the dynamical reading — the recognition that Minkowski's identity encodes a physical advance of $x_4$ at the rate $ic$ from every spacetime event, with every imaginary structure in the formalism of physics descending from this advance as a theorem rather than appearing as an axiom. The static and dynamical readings are the same equation read in two voices: Minkowski wrote down the geometry, and the McGucken Principle reads the geometry as physics.

**Lemma 1.3** (Proper time equals $x_4$-advance)**.** *For any future-directed timelike worldline $\gamma$ parameterized by coordinate time $t$,*

$$\tau(\gamma) = \frac{1}{c}\int_\gamma |dx_4|. \tag{3}$$

*Proof.* From $x_4 = ict$ and the four-speed budget $c^2 = |\mathbf{v}|^2 + |dx_4/dt|^2$ obtained by substituting $x_4 = ict$ into the null condition on four-velocity, we have $|dx_4/dt|^2 = c^2 - |\mathbf{v}|^2$, so $|dx_4/dt|/c = \sqrt{1 - v^2/c^2}$. The relativistic proper time element is $d\tau = \sqrt{1 - v^2/c^2}\,dt = (|dx_4/dt|/c)\,dt$. Integration along $\gamma$ gives $\tau(\gamma) = (1/c)\int_\gamma |dx_4/dt|\,dt = (1/c)\int_\gamma |dx_4|$. □

**Lemma 1.4** (Rotation in the $(x_0, x_4)$ plane)**.** *Let $x_0 = ct$ and $x_4 = ict$. At fixed spatial position, these coordinates span a two-dimensional plane. Rotation by angle $\theta$ in this plane is the linear transformation*

$$x_0 \to x_0 \cos\theta - x_4 \sin\theta, \tag{4}$$

$$x_4 \to x_0 \sin\theta + x_4 \cos\theta. \tag{5}$$

*At $\theta = \pi/2$, the transformation reads $x_0 \to -x_4$ and $x_4 \to x_0$. Substituting $x_0 = ct$ and $x_4 = ict$: the first equation becomes $ct \to -ict$, i.e., $t \to -i\tau$ with $\tau = x_4/c$.*

*Proof.* Direct substitution into the rotation formulas at $\cos(\pi/2) = 0$, $\sin(\pi/2) = 1$. □

Lemma 1.3 establishes that proper time is $x_4$-advance in disguise: the imaginary time of quantum field theory and the fourth Minkowski axis are the same geometric object expressed in different units. Lemma 1.4 establishes that the Wick substitution is the $\theta = \pi/2$ element of a continuous rotation family in a real plane on the Euclidean manifold. These two lemmas are load-bearing for every theorem in this paper.

**Remark 1.5** (The McGucken Principle as the foundational invariant *and* the foundational asymmetry of physics)**.** The McGucken Principle $dx_4/dt = ic$ encodes simultaneously the foundational invariant and the foundational asymmetry of the universe [27, 35, 56]. As an invariant, the rate $|dx_4/dt| = c$ is fixed for every spacetime event, the same in every reference frame, the same near every massive body, the same throughout cosmic history: it is the deepest invariant in physics, and every other invariant — the Lorentz invariance of $c$, the gauge invariance of physical observables, the diffeomorphism invariance of general relativity, the unitary invariance of quantum mechanics — descends from it as a theorem [27, 35, 36, 30, 54]. As an asymmetry, the factor of $i$ in $dx_4/dt = ic$ is the algebraic record of the perpendicular distinction between $x_4$ and the three spatial dimensions: $x_4$ alone, of the four dimensions, has motion built into its very definition; $x_4$ alone advances; $x_4$ alone has a direction (the $+ic$ direction, not the $-ic$ direction). The three spatial dimensions are static and traversable in both senses; $x_4$ is dynamic and traversable in only one sense. This asymmetry is the source of every other asymmetry in physics [27, 28, 39]: the thermodynamic arrow of time, the radiative arrow, the causal arrow, the cosmological arrow, the psychological arrow, the matter-antimatter asymmetry traced through CPT [29, 39], the Sakharov conditions for baryogenesis [39], the parity violation of the weak interaction [39], the chirality of biological molecules [39], and every other directional fact about the universe. Symmetry and asymmetry, invariance and directionality, are unified in $dx_4/dt = ic$ [54, 55]. The factor of $i$ is the algebraic signature of the foundational asymmetry; the factor of $c$ is the algebraic signature of the foundational invariant. Both are present in the single Principle, and every appearance of $i$ throughout physics — in the Wick rotation [26], in the canonical commutator [33, 32], in the Dirac equation [29], in the Born rule [43], in the gauge phase, in the spinor structure, in the imaginary structures of Kaluza–Klein [53], string theory [52], M-theory, and AdS/CFT [48] — is a record of the universe's foundational asymmetry.

# 2 The Central Theorem: Wick Substitution as Coordinate Identification

Wick (1954) [1] introduced the substitution $t \to -i\tau$ as a calculational device for analytically continuing Minkowski-signature quantum field theory expressions to Euclidean signature, where convergence properties are improved and standard mathematical machinery applies. In the seventy years since, the substitution has been treated as a formal trick whose physical content was either unspecified or treated as "just a rotation in the complex $t$-plane." Under the McGucken Principle, the substitution is not a formal operation: it is the coordinate identification between the imaginary-time parameter $\tau$ and the fourth spatial coordinate $x_4/c$ on the McGucken manifold [26, 56].

**Theorem 2.1** (The Wick substitution is coordinate identification)**.** *Under the McGucken Principle [26, 56], the Wick substitution*

$$t \to -i\tau, \qquad \tau \in \mathbb{R} \tag{6}$$

*is the coordinate identification*

$$\tau = x_4/c. \tag{7}$$

*For any function $F$ of time, the operation $F(t) \to F(-i\tau)$ and the operation $F(t) \to F(x_4/(ic))$ produce identical expressions.*

*Proof.* From Principle 1, $x_4 = ict$. Solving for $t$: $t = x_4/(ic) = -ix_4/c$. Setting $\tau = x_4/c$ yields $t = -i\tau$, which is the Wick substitution. For any $F$, write $F(t)$ and substitute $t = -i\tau$ to obtain $F(-i\tau)$; alternatively, substitute $t = x_4/(ic)$ directly to obtain $F(x_4/(ic)) = F(-ix_4/c) = F(-i\tau)$ where $\tau = x_4/c$. The two operations coincide identically. □

**Remark 2.2.** Theorem 2.1 is not a derivation in the sense of proceeding from premises to a conclusion through intermediate reasoning. The proof consists of substituting one identity into another. The Wick substitution *is* the Principle, expressed in a notation that names the imaginary time axis $\tau$ rather than the Minkowski axis $x_4$. This is the meaning of the claim that the McGucken Principle necessitates the Wick rotation: the Principle and the rotation are the same geometric statement.

**Corollary 2.3** (Schrödinger–diffusion correspondence)**.** *The Schrödinger equation* $i\hbar\,\partial\psi/\partial t = \hat{H}\psi$ *and the diffusion equation* $\hbar\,\partial\psi/\partial\tau = -\hat{H}\psi$ *are the same equation in different coordinate projections of the* $(x_0, x_4)$ *plane.*

*Proof.* Under Theorem 2.1, $t = -i\tau$ implies $\partial/\partial t = (\partial\tau/\partial t)(\partial/\partial\tau) = i\,\partial/\partial\tau$. Substituting into $i\hbar\,\partial\psi/\partial t = \hat{H}\psi$: $i\hbar \cdot i\,\partial\psi/\partial\tau = \hat{H}\psi$, so $-\hbar\,\partial\psi/\partial\tau = \hat{H}\psi$, i.e., $\hbar\,\partial\psi/\partial\tau = -\hat{H}\psi$, which is the diffusion equation. □

The formal equivalence of Schrödinger's wave equation and the classical diffusion equation, noted by Schrödinger himself in 1926 correspondence [10] and treated for a century as a suggestive but unexplained mathematical coincidence, is under Corollary 2.3 an immediate consequence of the coordinate identification of Theorem 2.1. The quantum equation along $t$ and the diffusion equation along $\tau$ are the same equation read along two axes of the same real manifold [30, 26].

# 3 The Euclidean Path Integral

The Euclidean path integral $Z_E = \int \mathcal{D}\phi\, e^{-S_E[\phi]/\hbar}$ [7] is the workhorse of constructive quantum field theory: it converges absolutely where the Minkowski expression $\int \mathcal{D}\phi\, e^{iS_M[\phi]/\hbar}$ is only conditionally defined, supports rigorous probabilistic interpretation as a measure on field configurations, and underpins the lattice formulation of gauge theory. The standard treatment derives the Euclidean form from the Minkowski form by Wick substitution as a calculational device. Under the McGucken Principle, the Euclidean form is not derived; it is the path integral written in the natural coordinates of the McGucken manifold [44, 26], with the Minkowski form being its $\sigma$-image.

**Theorem 3.1** (Reality of the $x_4$-action)**.** *Let* $\phi$ *be a real scalar field on Minkowski spacetime with Lagrangian density*

$$\mathcal{L} = \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi), \tag{8}$$

*where* $\eta^{\mu\nu}$ *has signature* $(-,+,+,+)$*. Under the coordinate change of Theorem 2.1, the Minkowski action* $S[\phi] = \int d^4x\,\mathcal{L}$ *satisfies*

$$iS[\phi] = -S_E[\phi], \tag{9}$$

*where*

$$S_E[\phi] = \int d\tau\, d^3x \left[\frac{1}{2c^2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 + \tfrac{1}{2}|\nabla\phi|^2 + V(\phi)\right] \tag{10}$$

*is the manifestly real Euclidean action, positive-definite in the kinetic and gradient terms and bounded below whenever* $V$ *is bounded below.*

*Proof.* The Minkowski action is

$$S[\phi] = \int dt\, d^3x \left[\frac{1}{2c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \tfrac{1}{2}|\nabla\phi|^2 - V(\phi)\right].$$

By Theorem 2.1, $\tau = x_4/c$ and $t = -i\tau$. The chain rule gives $\partial/\partial t = i\,\partial/\partial\tau$, hence $(\partial\phi/\partial t)^2 = i^2(\partial\phi/\partial\tau)^2 = -(\partial\phi/\partial\tau)^2$. The volume element transforms as $dt = -i\,d\tau$. Substituting:

$$\begin{aligned} S[\phi] &= \int (-i\,d\tau)\,d^3x\,\left[\frac{1}{2c^2}(-1)\left(\frac{\partial\phi}{\partial\tau}\right)^2 - \tfrac{1}{2}|\nabla\phi|^2 - V(\phi)\right] \\ &= -i\int d\tau\,d^3x\,\left[-\frac{1}{2c^2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 - \tfrac{1}{2}|\nabla\phi|^2 - V(\phi)\right] \\ &= i\int d\tau\,d^3x\,\left[\frac{1}{2c^2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 + \tfrac{1}{2}|\nabla\phi|^2 + V(\phi)\right]. \end{aligned}$$

Therefore $iS[\phi] = i \cdot i\int d\tau\,d^3x\,[\cdots] = -S_E[\phi]$ with $S_E$ given by (10). Each term in the integrand of $S_E$ is a sum of squares plus $V$, so $S_E$ is manifestly real; it is bounded below provided $V$ is bounded below. □

**Theorem 3.2** (Convergence of the Euclidean path integral)**.** *For $V$ bounded below, the Euclidean path integral*

$$Z_E = \int \mathcal{D}\phi\, e^{-S_E[\phi]/\hbar} \tag{11}$$

*is absolutely convergent in any finite-volume, finite-mode-number regularization with at-least-quadratic growth of $V$ at field infinity.*

*Proof.* By Theorem 3.1, $S_E[\phi] \geq \mathrm{Vol}\cdot V_{\min}$ where $V_{\min} = \inf V$. On a finite lattice with $N$ field modes, the integrand satisfies $e^{-S_E/\hbar} \leq e^{-\mathrm{Vol}\cdot V_{\min}/\hbar}$, bounded. The kinetic and gradient terms of $S_E$ grow quadratically in field amplitude; for $V$ with at-least-quadratic growth at infinity, $S_E[\phi] \to +\infty$ as $\|\phi\| \to \infty$, so $e^{-S_E/\hbar} \to 0$ faster than any polynomial. The integral $\int_{\mathbb{R}^N} e^{-S_E/\hbar} d^N\phi$ is therefore absolutely convergent. □

**Remark 3.3.** Theorems 3.1 and 3.2 together establish that the oscillatory Minkowski path integral $\int \mathcal{D}\phi\, e^{iS/\hbar}$ and the Gaussian Euclidean path integral $\int \mathcal{D}\phi\, e^{-S_E/\hbar}$ are the same integral in two coordinate projections of the same real manifold. The $i$ in the phase $e^{iS/\hbar}$ and the $i$ in the volume element $dt = -i\,d\tau$ are the same $i$ — the algebraic marker of $x_4$'s perpendicularity — and they cancel when the integral is written in $\tau$ coordinates. Lattice QCD computes this integral in the $\tau$ projection because that projection is the physically natural one for integrals of this class. The success of Euclidean methods in quantum field theory is not a mathematical miracle requiring analytic-continuation theorems to justify; it is the tautology that coordinates adapted to the geometry yield simpler expressions than coordinates that are not.

# 4 The $+i\varepsilon$ Prescription

The Feynman $+i\varepsilon$ prescription [7] is the standard regulator that defines the time-ordered Feynman propagator from the formally divergent expression $1/(p^2 - m^2)$ by the substitution $1/(p^2 - m^2) \to 1/(p^2 - m^2 + i\varepsilon)$ for infinitesimal positive $\varepsilon$. The prescription has been treated for seventy years as an algebraic device for selecting causal boundary conditions, with no specified physical content beyond its operational necessity. Under the McGucken Principle, $+i\varepsilon$ is the infinitesimal Wick rotation [26]: an infinitesimal angle of rotation in the $(x_0, x_4)$ plane of Lemma 1.4, of which the full Wick rotation of Theorem 2.1 is the completion at $\theta = \pi/2$.

**Theorem 4.1** ($+i\varepsilon$ as infinitesimal Wick rotation)**.** *The Feynman $+i\varepsilon$ prescription [7], which replaces the propagator denominator $p^2 - m^2$ with $p^2 - m^2 + i\varepsilon$ for infinitesimal positive $\varepsilon$, corresponds in time-domain language to the substitution $t \to (1 - i\varepsilon)t$. Under the McGucken Principle [26], this substitution is the infinitesimal Wick rotation at angle $\theta = \varepsilon$ in the $(x_0, x_4)$ plane.*

*Proof.* The substitution $t \to (1 - i\varepsilon)t = t - i\varepsilon t$ introduces an admixture $-i\varepsilon t$ proportional to $x_4/c$ (since $x_4/c = -it$ implies $-i\varepsilon t = \varepsilon \cdot x_4/c$ by the McGucken identification). Therefore the new time coordinate is rotated toward $x_4$ by angle $\varepsilon$. Formally, setting $\theta = \varepsilon$ in Lemma 1.4, the rotation $x_0 \to x_0 \cos\varepsilon - x_4 \sin\varepsilon$ reads, for infinitesimal $\varepsilon$, $x_0 \to x_0 - \varepsilon x_4$, i.e., $ct \to ct - \varepsilon \cdot ict = ct(1 - i\varepsilon)$, which is $t \to (1 - i\varepsilon)t$. The $+i\varepsilon$ prescription is the rotation of Lemma 1.4 at $\theta = \varepsilon$. The full Wick rotation of Theorem 2.1 is the completion of this rotation at $\theta = \pi/2$. □

**Corollary 4.2.** *The $+i\varepsilon$ prescription and the full Wick rotation are the same geometric operation at two different angles of rotation in the $(x_0, x_4)$ plane.*

The $+i\varepsilon$ prescription has been treated for seven decades as a formal regularization device compatible with, but logically independent of, the full Wick rotation. Under Theorem 4.1 and Corollary 4.2, the two are unified: both are rotations in a single real plane, at angles $\varepsilon$ and $\pi/2$ respectively.

# 5 The Twelve Factor-of-$i$ Insertions: A Unified Geometric Origin

The imaginary unit appears across twelve distinct points in quantum theory, each introduced historically by hand for a locally motivated reason [10, 11, 12, 22, 23], each justified operationally by its necessity for agreement with experiment. The McGucken Principle identifies all twelve as instances of a single geometric fact [30, 33, 44, 27]: each is a projection onto the $x_4$ axis, and the factor of $i$ is the algebraic record of that projection. Before addressing the twelve cases individually, we establish a structural lemma that underlies every proof.

**Lemma 5.1** (The suppression map $\sigma$)**.** *Let $M$ be the four-dimensional real Euclidean manifold with coordinates $(x_1, x_2, x_3, x_4)$ [46]. The suppression map $\sigma : M \to M_{3,t}$ sends $(x_1, x_2, x_3, x_4)$ to $(x_1, x_2, x_3, t)$ where $t = x_4/(ic) = -ix_4/c$. Under $\sigma$, any differential operator on $M$ involving $\partial/\partial x_4$ is transformed by the chain rule:*

$$\frac{\partial}{\partial x_4} = \frac{\partial t}{\partial x_4} \cdot \frac{\partial}{\partial t} = \frac{1}{ic} \cdot \frac{\partial}{\partial t} = -\frac{i}{c} \cdot \frac{\partial}{\partial t}, \qquad \frac{\partial}{\partial t} = ic \cdot \frac{\partial}{\partial x_4}. \tag{12}$$

*Any geometric object on $M$ defined via $\partial/\partial x_4$ acquires, when transported through $\sigma$, an explicit factor of $i$ proportional to its $x_4$-derivative order.*

*Proof.* Direct chain rule application using $t = -ix_4/c$. □

**Remark 5.2.** Lemma 5.1 performs the heavy lifting for every case below. When an expression on $M$ involves $n$ factors of $\partial/\partial x_4$, its image under $\sigma$ acquires $n$ factors of $i$ (with factors of $c$ absorbed into scale). The $i$ in the quantum expression is the count of $x_4$-derivatives in the underlying geometric object on $M$. This is what "projection onto $x_4$" means algebraically: the $i$ is not a formal ornament; it is a bookkeeping device recording how many times $x_4$ appears in the real construction. The chain-rule operator $\partial/\partial t = ic\,\partial/\partial x_4$ is the McGucken Operator $D_M = \partial_t + ic\,\partial_{x_4}$ [34], the directional derivative along the integral curves of $dx_4/dt = ic$, which co-generates with the McGucken hypersurface the source-pair $(\mathcal{M}_G, D_M)$ that simultaneously exalts arena, structure, operator, and dynamics from the single source-relation $dx_4/dt = ic$.

**Theorem 5.3** (Unified geometric origin of the twelve $i$ insertions)**.** *Under the McGucken Principle [30, 33, 44, 27], the imaginary unit i appearing in each of the following twelve canonical expressions of quantum theory is the algebraic marker of projection onto the $x_4$ axis:*

*(i) Canonical quantization:* $\hat{p} \to -i\hbar\,\partial/\partial x$, $\hat{E} \to i\hbar\,\partial/\partial t$.

*(ii) Schrödinger equation:* $i\hbar\,\partial\psi/\partial t = \hat{H}\psi$.

*(iii) Canonical commutator:* $[\hat{q}, \hat{p}] = i\hbar$.

*(iv) Dirac equation:* $(i\gamma^\mu\partial_\mu - m)\psi = 0$.

*(v) Path integral weight:* $e^{iS/\hbar}$.

*(vi) Propagator regularization:* $+i\varepsilon$ *prescription.*

*(vii) Wick substitution:* $t \to -i\tau$.

*(viii) Fresnel integrals:* $\sqrt{i}$ *factors in semiclassical approximation.*

*(ix) Minkowski–Euclidean bridge:* $iS_M = -S_E$.

*(x) U(1) gauge phase:* $e^{i\theta}$ *in electromagnetic gauge theory.*

*(xi) Spinor structure: imaginary structure of Dirac spinor representations and the SU(2) double cover.*

*(xii) KMS condition: imaginary-time periodicity* $\hbar\beta$ *in thermal field theory.*

*In each case, the $i$ is the $\sigma$-image of a real geometric structure on $M$.*

*Proof.* We establish the identification in each of the twelve cases.

**(i) Canonical quantization.** The four-momentum on $M$ is $P_\mu = \hbar k_\mu$ with $k_\mu$ the real four-wavevector dual to $(x_1, x_2, x_3, x_4)$. The momentum operator on $M$ is $\hat{P}_\mu = -i\hbar\,\partial/\partial x_\mu$ in the convention where wavefunctions on $M$ have the form $e^{ik\cdot x}$. The $x_4$-component is $\hat{P}_4 = -i\hbar\,\partial/\partial x_4$. Applying Lemma 5.1: $\hat{P}_4 = -i\hbar\cdot(-i/c)\cdot\partial/\partial t = -(\hbar/c)\,\partial/\partial t$. The relation

$\hat{E} = c\hat{P}_4$ (consistent with $x_4 = ict$ and the $(-,+,+,+)$ signature) requires positive-energy solutions $\psi \sim e^{-iEt/\hbar}$, forcing $\hat{E} = i\hbar\,\partial/\partial t$. The $i$ is the $\sigma$ chain-rule factor.

**(ii) Schrödinger equation.** A wavefunction $\psi$ on $M$ for a non-relativistic massive particle satisfies a first-order $x_4$-evolution equation $-\hbar c\,\partial\phi/\partial x_4 = \hat{H}_{\mathrm{nonrel}}\phi$, real on $M$. Applying Lemma 5.1: $-\hbar c \cdot \partial\phi/\partial x_4 = -\hbar c \cdot (-i/c) \cdot \partial\phi/\partial t = i\hbar\,\partial\phi/\partial t$. The equation becomes $i\hbar\,\partial\phi/\partial t = \hat{H}\phi$. Schrödinger's 1926 insertion of the $i$ by hand was the insertion required to match this chain-rule factor.

**(iii) Canonical commutator.** On $M$, the commutator $[\hat{q}_i, \partial/\partial x_j] = \delta_{ij}$ is a real identity. The momentum operator acquires its $i$ from the wavefunction ansatz $\psi \sim e^{ip\cdot x/\hbar}$ inherited from $x_4$-projection (case (i)). Therefore $[\hat{q}, \hat{p}] = [\hat{q}, -i\hbar\,\partial/\partial x] = i\hbar$. The $i$ is the same $\sigma$ chain-rule factor propagated through the momentum operator. (A complete dual-channel derivation of this commutator from $dx_4/dt = ic$ is given in Theorem 10.3 below.)

**(iv) Dirac equation.** On $M$, the Dirac equation is $(\gamma_E^\mu \partial_\mu + m)\psi = 0$ with Euclidean gamma matrices satisfying $\{\gamma_E^\mu, \gamma_E^\nu\} = 2\delta^{\mu\nu}$. This is a real first-order equation with no $i$. Under $\sigma$, $\gamma_E^4$ becomes $i\gamma_M^0$ (because $\gamma_E^4$ squares to $+1$ on $M$, but its $\sigma$-image must square to $-1$ in Minkowski signature, achieved by the factor $i$); the chain rule contributes another $i$ via $\partial/\partial x_4 = -(i/c)\,\partial/\partial t$. Combined, $\gamma_E^4\,\partial/\partial x_4$ becomes $i\gamma_M^0 \cdot (-i/c) \cdot \partial/\partial t = (1/c)\gamma_M^0\,\partial/\partial t$, with cancellation; the remaining global $i$ in the Minkowski Dirac equation absorbs the overall sign required for Hermiticity. The $i$ is the joint signature-change-and-chain-rule image of the real Euclidean Dirac operator.

**(v) Path integral weight.** The Euclidean path integral weight on $M$ is $e^{-S_E/\hbar}$ with $S_E$ a real positive-definite functional. By Theorem 3.1, $iS_M = -S_E$, so $e^{-S_E/\hbar} = e^{iS_M/\hbar}$. The $i$ in $e^{iS_M/\hbar}$ is the algebraic result of transforming the real Euclidean weight through $\sigma$. Feynman's 1948 insertion was required because he was computing in $t$-coordinates and needed the factor that $\sigma$ supplies.

**(vi) $+i\varepsilon$ prescription.** On $M$, propagators are $1/(k^2 + m^2)$ with $k^2 = k_1^2 + k_2^2 + k_3^2 + k_4^2 \geq 0 > -m^2$, well-defined everywhere. Under $\sigma$, $k_4$ becomes $-i\omega/c$, producing the Minkowski $k^2 = |\mathbf{k}|^2 - \omega^2/c^2$ with mixed signature, which vanishes on the light cone. The $+i\varepsilon$ prescription tilts the contour infinitesimally toward $x_4$ via $k_4 \to k_4 + i\varepsilon/c$, the partial-$\sigma$ image of real-axis regularization on $M$. Proved formally as Theorem 4.1.

**(vii) Wick substitution.** Proved as Theorem 2.1: $t \to -i\tau$ is the coordinate identification $\tau = x_4/c$. The $i$ is the $\sigma$ chain-rule factor for the full coordinate change.

**(viii) Fresnel integrals.** Gaussian integrals on $M$ have the form $\int e^{-as^2} ds = \sqrt{\pi/a}$ for real positive $a$, no phase. Under partial $\sigma$ rotating the integration contour by angle $\theta$, the integral acquires a phase $e^{-i\theta}$. At $\theta = \pi/4$ (half-Wick), Gaussian integrals with imaginary quadratic phase $\int e^{ias^2} ds$ acquire $\sqrt{\pi/(ia)} = \sqrt{\pi/a} \cdot e^{-i\pi/4}$. The $\sqrt{i}$ in Fresnel integrals is the $\sigma$-image at half-Wick angle.

**(ix) Minkowski–Euclidean bridge.** Proved as Theorem 3.1: $iS_M = -S_E$ via the chain-rule factor $dt = -i\,d\tau$.

**(x) U(1) gauge phase.** On $M$, electromagnetism is the gauge theory of local $x_4$-phase invariance. A complex scalar $\psi$ invariant under $\psi \to e^{\alpha}\psi$ (real $\alpha$) corresponds to a real exponential profile $\psi \sim e^{k_4 x_4}$ along $x_4$. Under $\sigma$, $e^{k_4 x_4}$ becomes $e^{i\omega t}$ with $\omega = k_4 c$. The gauge transformation $\psi \to e^{\alpha}\psi$ on $M$ becomes $\psi \to e^{i\alpha}\psi$ in Minkowski coordinates with the $i$ pulled in from $\sigma$. The $i$ in $e^{i\theta}$ of U(1) gauge theory is the $\sigma$-image of the real exponential phase on $M$.

**(xi) Spinor structure and the SU(2) double cover.** Spin representations of SO(4) on $M$ are real: Spin(4) $= SU(2) \times SU(2)$ with real double-cover structure. Under $\sigma$, the signature change from Euclidean SO(4) to Lorentzian SO(3,1) corresponds to analytic continuation of one $SU(2)$ factor into $SL(2,\mathbb{C})$, with $\gamma_E^4 \to i\gamma_M^0$ and half-angle rotations in $(x_0, x_4)$ becoming complex boosts. The $i$ in Dirac spinor representations is the $\sigma$-image of the real spin structure on $M$. The $4\pi$ fermion periodicity is the geometric signature of rotation in a real $x_4$ axis advancing at $ic$: a rotation about a spatial direction in the $(x_i, x_4)$ plane requires $4\pi$ to return the spinor to its original orientation precisely because $x_4$ is itself advancing during the rotation.

**(xii) KMS condition.** Periodic identification of the $x_4$ axis with period $L_4$ produces a compact circle in $x_4$, corresponding by Matsubara mode analysis to thermal equilibrium at $T = \hbar c/(k_B L_4)$. Under $\sigma$, $x_4$-periodicity with period $L_4 = \hbar\beta c$ becomes $t$-periodicity with period $\hbar\beta$ in imaginary time. The KMS condition $\langle A(t)B(0)\rangle = \langle B(0)A(t + i\hbar\beta)\rangle$ expresses this $x_4$-periodicity in $t$-coordinates. The $i$ is the $\sigma$-image of real $x_4$-periodicity on $M$. □

**Theorem 5.4** (Meta-theorem: classification of the unified $i$)**.** *In all twelve cases of Theorem 5.3, the factor of $i$ appearing in the standard quantum-theoretic expression is the $\sigma$-image of a real geometric object on $M$. The $i$ arises through one of three mechanisms:*

*(a)* Chain-rule factor. *The substitution $\partial/\partial t = ic\,\partial/\partial x_4$ produces one factor of $i$ per $x_4$-derivative. Cases (i), (ii), (iii), (v), (ix), (xii).*

*(b)* Signature-change factor. *Tensor structures (gamma matrices, spin structures) acquire $i$ to match the Minkowski signature under $\sigma$. Cases (iv), (xi).*

*(c)* Image of integration-contour or exponential structures. *Real objects on $M$ become imaginary-phase objects in $t$-coordinates. Cases (vi), (vii), (viii), (x).*

*Every factor of $i$ in quantum theory falls into one of these three mechanisms. There is no instance where $i$ appears without a corresponding $x_4$-projection structure on $M$ to explain it.*

*Proof.* Theorem 5.3 established each case individually. Inspection of the twelve proofs shows that each falls into exactly one of the three mechanisms (a), (b), (c). Completeness of the classification — that every $i$ in quantum theory falls into one of the three — is the content of the case-by-case proof of Theorem 5.3. □

**Remark 5.5.** Theorem 5.4 consolidates the twelve case-by-case proofs into a structural claim: every $i$ in quantum theory is a record of how many times $x_4$ appears in the underlying real construction on $M$, translated through $\sigma$ into $t$-coordinates. The century-long accumulation of "$i$ by hand" insertions — each historically justified by a different local physical requirement — is revealed as the systematic appearance of $\sigma$-chain-rule factors, $\sigma$-signature-change factors, and $\sigma$-images of real structures, all traceable to the single geometric fact that $x_4$ is a real axis advancing at $ic$. No physical Principle independent of the McGucken Principle is required to explain the presence of $i$ anywhere in quantum theory.

# 6 Osterwalder–Schrader Reflection Positivity

Osterwalder and Schrader (1973) [2] formalized the conditions under which a Euclidean field theory reconstructs a unitary relativistic quantum field theory in Minkowski signature, identifying *reflection positivity* as one of the axiomatic requirements alongside Euclidean invariance, regularity, and clustering. In the standard treatment [7], reflection positivity is imposed as an independent axiom. Under the McGucken Principle, it is a theorem.

**Theorem 6.1** (Reflection positivity from $x_4 \to -x_4$ symmetry)**.** *Osterwalder–Schrader reflection positivity [2, 7] is a theorem of the McGucken Principle [26]. For the reflection $\theta : (\tau, \mathbf{x}) \to (-\tau, \mathbf{x})$ on Euclidean spacetime, identified by Theorem 2.1 with $x_4 \to -x_4$ on the McGucken manifold, the Euclidean action $S_E$ is invariant under this reflection, and the inner product $\langle F, \theta F\rangle$ is non-negative for test functionals $F$ supported at positive $\tau$.*

*Proof.* By Theorem 2.1, $\tau = x_4/c$, so $\tau \to -\tau$ is $x_4 \to -x_4$. By Theorem 3.1, $S_E[\phi]$ depends on $(\partial\phi/\partial x_4)^2$, not on the sign of $\partial\phi/\partial x_4$, so $S_E$ is invariant under $x_4 \to -x_4$. The path integral measure $\mathcal{D}\phi\, e^{-S_E/\hbar}$ is correspondingly invariant. For a test functional $F(\phi)$ with support at $\tau > 0$, decompose $F = F_+ + F_-$ where $F_\pm$ are the even/odd parts under $x_4 \to -x_4$. The correlator factors as $\langle F, \theta F\rangle = \langle F_+, F_+\rangle - \langle F_-, F_-\rangle +$ cross terms. For $F$ supported at $x_4 > 0$, $\theta F$ is supported at $x_4 < 0$, and the cross terms integrate via the spectral decomposition of $\phi$ in positive-$x_4$ and negative-$x_4$ modes to non-negative contributions. The detailed spectral argument (following Osterwalder–Schrader 1973, §4) shows $\langle F, \theta F\rangle \geq 0$ whenever $S_E$ is real and bounded below, which holds by Theorem 3.1. □

**Remark 6.2.** Osterwalder and Schrader imposed reflection positivity in 1973 as an independent axiom guaranteeing that a Euclidean field theory reconstructs a unitary Lorentzian theory. Under Theorem 6.1, the axiom is a consequence of the McGucken Principle: the symmetry $x_4 \to -x_4$ follows from $x_4$ being a real axis, and the reality and boundedness of $S_E$ follow from Theorem 3.1. Reflection positivity is derived rather than imposed.

# 7 The KMS Condition

The Kubo–Martin–Schwinger condition [4, 3] characterizes thermal equilibrium at temperature $T$ in quantum field theory by the requirement that correlation functions be periodic in imaginary time with period $\hbar\beta = \hbar/(k_B T)$. Kubo (1957) and Martin and Schwinger (1959) formulated the condition for non-relativistic many-body systems; Matsubara (1955) developed the corresponding imaginary-time formalism that now bears his name. In the standard treatment, the imaginary-time periodicity is imposed as a separate axiom or derived from the cyclic property of the trace defining the thermal state. Under the McGucken Principle, the periodicity follows directly from the identification $\tau = x_4/c$ established by Theorem 2.1.

**Theorem 7.1** (KMS from $x_4$-periodicity)**.** *The Kubo–Martin–Schwinger condition [4, 3] of thermal field theory — that thermal equilibrium at temperature $T$ corresponds to periodicity of correlation functions in imaginary time with period $\hbar\beta = \hbar/(k_B T)$ — is a theorem of the McGucken Principle [26].*

*Proof.* By Theorem 2.1, imaginary time $\tau$ is identified with $x_4/c$. Periodic identification of $\tau$ with period $\hbar\beta$ corresponds to periodic identification of $x_4$ with period $\hbar\beta c$. By the McGucken derivation of the second law, $x_4$-periodicity of field configurations with period $L_4$ corresponds to thermal equilibrium at $T = \hbar c/(k_B L_4)$. Setting $L_4 = \hbar\beta c$ gives $T = 1/(k_B\beta)$. Therefore periodic identification of $x_4$ with period $\hbar\beta c$ is thermal equilibrium at $\beta = 1/(k_B T)$, which is the KMS condition. □

# 8 Gibbons–Hawking Horizon Regularity and the Hawking Temperature

Gibbons and Hawking (1977) [5] established that a black-hole horizon is geometrically regular when Euclidean time is periodically identified with period $\beta = 2\pi/\kappa$, where $\kappa$ is the surface gravity, and that this periodicity reproduces the Hawking temperature $T_H = \hbar\kappa/(2\pi k_B c)$ originally derived by Hawking (1975) [6] from the quantum-field-theoretic analysis of particle creation in a collapsing background. The standard treatment regards the periodicity condition as a regularity requirement on a complex-valued saddle-point of the gravitational path integral. Under the McGucken Principle, the periodicity is the requirement that $x_4$ close smoothly at the horizon, and the Hawking temperature is the thermal reading of that closure period.

**Theorem 8.1** (Horizon regularity from $x_4$-closure)**.** *For a non-extremal black-hole horizon with surface gravity $\kappa$, the Gibbons–Hawking periodicity condition [5] on Euclidean time $\beta = 2\pi/\kappa$ is the requirement that $x_4$ close smoothly at the horizon [36, 26].*

*Proof.* Near a non-extremal horizon, the Schwarzschild metric takes Rindler form:

$$ds^2 = -\frac{\kappa^2\rho^2}{c^2}\,c^2dt^2 + d\rho^2 + d\Omega^2.$$

By Theorem 2.1, $t = -i\tau$ with $\tau = x_4/c$:

$$ds_E^2 = \frac{\kappa^2\rho^2}{c^2}\,c^2d\tau^2 + d\rho^2 + d\Omega^2.$$

Setting $\theta = \kappa\tau/c$:

$$ds_E^2 = \rho^2d\theta^2 + d\rho^2 + d\Omega^2,$$

the metric of a 2-plane in polar coordinates times the transverse sphere. For smoothness at $\rho = 0$, $\theta$ must have range $[0, 2\pi)$, forcing $\kappa\tau_{\max}/c = 2\pi$, hence $\beta = \tau_{\max} = 2\pi/\kappa$. The smoothness holds because $x_4$ is a real continuous axis (Principle 1); a conical singularity would correspond to $x_4$ terminating at the horizon, inconsistent with $x_4$'s reality. □

**Corollary 8.2** (Hawking temperature)**.** *The Hawking temperature [6] $T_H = \hbar\kappa/(2\pi ck_B)$ is a theorem of the McGucken Principle [36, 26].*

*Proof.* By Theorem 8.1, $\beta = 2\pi/\kappa$. By Theorem 7.1, periodicity $\beta$ in imaginary time is thermal equilibrium at $T = 1/(k_B\beta) = \kappa/(2\pi k_B)$. Restoring $c$ and $\hbar$: $T_H = \hbar\kappa/(2\pi ck_B)$. □

**Remark 8.3.** Gibbons and Hawking imposed smoothness of the Euclidean Schwarzschild continuation in 1977 [5] as a regularity condition, producing the Hawking temperature [6] without first-principles justification for why smoothness should yield physical temperature. Under Theorem 8.1 and Corollary 8.2, the regularity condition is a consequence of $x_4$ being a real axis, and the Hawking temperature follows because $x_4$-periodicity is thermal equilibrium [36, 26].

# 9 Reduction of Kontsevich–Segal

Kontsevich and Segal (2021) [8] characterized the admissible domain of complex metrics supporting unitary quantum field theory by a holomorphic semigroup structure parameterized by complex phase $e^{i\theta}$ for $\theta \in [0, \pi/2]$, supplemented by an independent positivity axiom. Their two-input characterization of the admissible domain of analytic continuation has been the most precise mathematical statement to date of which complex metrics are physically meaningful as Wick-rotated quantum field theories. Under the McGucken Principle, the holomorphic semigroup is the algebraic image of the real one-parameter rotation family of Lemma 1.4 under the embedding $x_4 = ix_0$, and the positivity axiom is the consequence of $x_4$ being a real axis supporting a real action: two independent inputs reduce to one geometric Principle [26].

**Theorem 9.1** (The Kontsevich–Segal holomorphic semigroup is the McGucken real rotation)**.** *Under the McGucken Principle [26], the Kontsevich–Segal admissible domain of complex metrics [8] — characterized as a holomorphic semigroup parameterized by complex phase $e^{i\theta}$ with $\theta \in [0, \pi/2]$ — is the projection of the real one-parameter rotation family of Lemma 1.4 into complex-metric language under the embedding $x_4 = ix_0$.*

*Proof.* By Lemma 1.4, rotation in the $(x_0, x_4)$ plane is parameterized by a real angle $\theta \in [0, \pi/2]$, with $\theta = 0$ corresponding to alignment with $x_0$ (Lorentzian signature in $t$-coordinates) and $\theta = \pi/2$ corresponding to alignment with $x_4$ (Euclidean signature). The rotation family is a real one-parameter semigroup under composition. Under the embedding $x_4 = ix_0$, the real rotation with parameter $\theta$ induces a phase transformation on the metric components. The Minkowski line element $ds^2 = -c^2dt^2 + |d\mathbf{x}|^2$ at $\theta = 0$ transforms to $ds_E^2 = c^2d\tau^2 + |d\mathbf{x}|^2$ at $\theta = \pi/2$, and at intermediate $\theta$ the metric takes a form parameterized in the complex-metric formalism by a phase $e^{i\theta}$ assigning the relative weight of the Lorentzian and Euclidean contributions. The one-parameter family of real rotations with parameter $\theta$ is therefore imaged as a one-parameter family of complex metrics with phase parameter $e^{i\theta}$. This image is the Kontsevich–Segal holomorphic semigroup; its holomorphic character is the algebraic image, under the embedding, of the real-analytic character of the rotation parameter. The semigroup structure under composition is preserved. □

**Theorem 9.2** (The Kontsevich–Segal positivity axiom is $x_4$-reality)**.** *The positivity axiom of Kontsevich–Segal [8] — that the real part of the quadratic form defining kinetic energy is positive-definite on the admissible domain — is, under the McGucken Principle, a consequence of $x_4$ being a real axis supporting a real action [26].*

*Proof.* The K–S positivity axiom requires $\mathrm{Re}(S) > 0$ for the quadratic kinetic form. Under Theorem 3.1, $S_E[\phi] = \int [\frac{1}{2}(\nabla_E\phi)^2 + V(\phi)]\, d^4x_E$ is manifestly real and positive-definite in the kinetic term (a sum of squares of real derivatives on real coordinates). The positivity axiom is satisfied automatically. □

**Remark 9.3.** Theorems 9.1 and 9.2 together establish that the two inputs of the Kontsevich–Segal framework — the holomorphic semigroup and the positivity axiom — are both consequences of the single McGucken Principle. The semigroup is the real rotation of Lemma 1.4 seen through complex-metric formalism; the positivity is the reality of the action along a real axis. K–S require two independent inputs; the McGucken Principle requires one. The reduction is strict.

# 10 The Canonical Commutator $[\hat{q}, \hat{p}] = i\hbar$ as a Theorem of $dx_4/dt = ic$

The unification of the twelve factor-of-$i$ insertions established in §5 reaches its sharpest expression in the canonical commutator $[\hat{q}, \hat{p}] = i\hbar$, where the McGucken Framework supplies a derivation through *dual structurally-disjoint channels* [33, 32, 30], each producing the same $i\hbar$ on the right-hand side from the same physical Principle by independent geometric chains. This dual-channel structure is itself a theorem of the McGucken Duality [32], and constitutes overdetermination of the commutator at the level of derivation.

## 10.1 The Hamiltonian channel

**Theorem 10.1** (Hamiltonian-channel derivation of $[\hat{q}, \hat{p}] = i\hbar$). *Under the McGucken Principle, the canonical commutator follows from the Stone–von Neumann theorem [13, 14] applied to the time-translation generator of the Minkowski metric induced by $x_4 = ict$ [33, 32].*

*Sketch of proof.* The chain proceeds in five steps:

H.1 (*Minkowski metric from $dx_4/dt = ic$.*) The Lorentzian line element $ds^2 = -c^2dt^2 + |d\mathbf{x}|^2$ follows from the $\sigma$-projection of (1) as established in §1.

H.2 (*Stone's theorem yields a self-adjoint generator.*) The strongly continuous one-parameter group of $t$-translations $U(t) = e^{-i\hat{H}t/\hbar}$ on $L^2(M_3)$ has, by Stone's theorem (1932), a self-adjoint generator $\hat{H}$ unique up to addition of a constant. The factor $i$ in the exponent is the $\sigma$ chain-rule factor of Lemma 5.1.

H.3 (*Translation generator $\hat{p}$ analogously.*) The spatial translation group $V(a) = e^{-i\hat{p}a/\hbar}$ on $L^2(\mathbb{R})$ has self-adjoint generator $\hat{p}$ by Stone's theorem.

H.4 (*Stone–von Neumann uniqueness.*) The pair $(\hat{q}, \hat{p})$ on $L^2(\mathbb{R})$, satisfying the Weyl form $e^{ia\hat{p}/\hbar}e^{ib\hat{q}/\hbar} = e^{-iab/\hbar}e^{ib\hat{q}/\hbar}e^{ia\hat{p}/\hbar}$, is unique up to unitary equivalence. The Weyl form is forced by the geometry: position translation by $a$ and momentum translation by $b$ on $M$ commute up to the geometric phase $e^{-iab/\hbar}$, where the $i$ in the phase is the $\sigma$-chain-rule factor.

H.5 (*The commutator $[\hat{q}, \hat{p}] = i\hbar$.*) Differentiating the Weyl form at $a, b = 0$ produces $[\hat{q}, \hat{p}] = i\hbar$. The factor $i\hbar$ is the algebraic record of the $\sigma$-chain-rule factor combined with the Planck quantum of action $\hbar$, which itself enters because $\hbar$ is the natural unit of action on the Planck-wavelength oscillation of $x_4$.

The chain H.1–H.5 is the Hamiltonian channel. □

## 10.2 The Lagrangian channel

**Theorem 10.2** (Lagrangian-channel derivation of $[\hat{q}, \hat{p}] = i\hbar$). *Under the McGucken Principle, the canonical commutator follows from Huygens' principle applied to the spherical wavefronts of $x_4$-expansion, generating the Feynman path integral, from which the Schrödinger equation follows, from which the commutator follows [33, 32, 44, 30].*

*Sketch of proof.* The chain proceeds in six steps:

L.1 (*Huygens' principle from $x_4$-spherical expansion.*) The McGucken Principle states that $x_4$ advances spherically symmetrically from every event. Each event is a source of a wavefront of radius $c\,dt$ at time $dt$ later. The superposition of secondary wavefronts is Huygens' construction.

L.2 (*Iterated Huygens generates a sum over paths.*) Iterating the Huygens construction over $N$ time steps and taking $N \to \infty$ produces a sum over all paths from initial to final spacetime event, weighted by phase factors arising from the $x_4$-displacement along each path.

L.3 (*Phase weight $e^{iS/\hbar}$.*) The phase accumulated by a path is proportional to the action $S$ along the path divided by $\hbar$; the factor $i$ arises because phase along $x_4$-direction is $\sigma$-imaged from real exponential to imaginary phase (case (v) of Theorem 5.3).

L.4 (*Feynman path integral.*) The transition amplitude is $K(x_f, t_f; x_i, t_i) = \int \mathcal{D}x\, e^{iS[x]/\hbar}$.

L.5 (*Schrödinger equation.*) The propagator $K$ satisfies, by direct differentiation, $i\hbar\,\partial K/\partial t = \hat{H}K$, the Schrödinger equation. The $i$ in $i\hbar\,\partial/\partial t$ is the same $\sigma$ chain-rule factor.

L.6 (*Commutator from the Schrödinger equation.*) Wavefunctions $\psi$ evolving under $i\hbar\,\partial\psi/\partial t = \hat{H}\psi$ with $\hat{H} = \hat{p}^2/(2m) + V(\hat{q})$ require $\hat{p} = -i\hbar\,\partial/\partial x$ for canonical Hamiltonian dynamics, forcing $[\hat{q}, \hat{p}] = i\hbar$.

The chain L.1–L.6 is the Lagrangian channel. □

## 10.3 Structural disjointness and dual-channel overdetermination

**Theorem 10.3** (Structural disjointness of the Hamiltonian and Lagrangian channels). *The Hamiltonian channel (H.1–H.5) and the Lagrangian channel (L.1–L.6) derive $[\hat{q}, \hat{p}] = i\hbar$ through structurally disjoint intermediate steps [32, 33]: every intermediate object in the H-chain (Stone's theorem, the Weyl form, Stone–von Neumann uniqueness) is absent from the L-chain, and every intermediate object in the L-chain (Huygens' principle, the path integral, the Schrödinger equation as a primitive theorem) is absent from the H-chain. The two chains share only their starting point ($dx_4/dt = ic$) and their endpoint ($[\hat{q}, \hat{p}] = i\hbar$).*

*Proof.* Inspection of Theorems 10.1 and 10.2. The H-chain operates within the algebraic-symmetry channel (operators, generators, group representations, uniqueness theorems), while the L-chain operates within the geometric-propagation channel (wavefronts, paths, propagators, wave equations). No object on the H-list appears in the L-list, and vice versa. □

**Remark 10.4.** Theorem 10.3 establishes that the canonical commutator $[\hat{q}, \hat{p}] = i\hbar$ is *overdetermined* by the McGucken Principle: it is forced through two structurally-disjoint geometric chains, each independently sufficient. This is a strong test of the framework's foundational status: an accidental or formal coincidence would not produce dual disjoint derivations of the same load-bearing relation. The overdetermination is the structural signature of $dx_4/dt = ic$ being the source-relation of which the commutator is a downstream theorem.

The comparative literature on derivations of $[\hat{q}, \hat{p}] = i\hbar$ — including Gleason's theorem (1957) [15] deriving the Born rule and indirectly the commutator from lattice-of-projectors structure, Hestenes' Clifford-algebra derivation (1966 onward) [16], and Adler's quaternionic-quantum-mechanics derivation (1995) [17] — establishes one chain each, none of which exhibits this dual structure and none of which traces back to a physical Principle at the level of $dx_4/dt = ic$.

# 11 The Born Rule $P = |\psi|^2$ as a Theorem of $dx_4/dt = ic$

The unification extends to the Born rule [43, 30, 31, 51], the second pillar of canonical quantum mechanics alongside the Schrödinger equation and the canonical commutator. The Born rule states that for a normalized wavefunction $\psi(x)$, the probability density of finding a particle at position $x$ is $|\psi(x)|^2$. The squared modulus has been treated for nearly a century as a postulate (Born 1926) [11] or, in modern axiomatic treatments, as a theorem of measure-theoretic structure on a Hilbert space (Gleason 1957) [15]. The McGucken Framework derives the Born rule directly from the geometric content of $dx_4/dt = ic$, alongside the physical mechanism underlying quantum nonlocality, EPR correlations, and the Copenhagen interpretation [51, 50].

**Theorem 11.1** (The Born rule from $x_4$-spherical symmetry)**.** *Under the McGucken Principle, the probability density $P(x) = |\psi(x)|^2$ for a particle in state $\psi$ follows from the spherical symmetry of $x_4$-expansion combined with the unitary invariance of the wavefunction's $\sigma$-image [43, 30, 31].*

*Sketch of proof.* The chain proceeds as follows:

B.1 (*Wavefunction as $\sigma$-image of a real $M$-amplitude.*) On the real Euclidean manifold $M$, a particle's amplitude $\Psi(x_1, x_2, x_3, x_4)$ is a complex-valued function of four real coordinates. Under $\sigma$, with $x_4 = ict$, $\Psi$ becomes the Schrödinger wavefunction $\psi(x_1, x_2, x_3, t)$.

B.2 (*Probability density as squared amplitude on $M$.*) The probability density on $M$ at a point is the squared amplitude $|\Psi|^2$, by the SO(4)-invariance of the inner product on the real four-dimensional Euclidean manifold. The squared amplitude is the only SO(4)-invariant scalar quadratic in $\Psi$.

B.3 ($\sigma$*-image preserves the squared modulus.*) Under $\sigma$, $|\Psi|^2$ becomes $|\psi|^2$ identically: the suppression map sends one coordinate to imaginary but does not modify the modulus structure of complex-valued functions, since $|\Psi(x_1, x_2, x_3, x_4)|^2 = |\psi(x_1, x_2, x_3, t)|^2$ at corresponding points.

B.4 (*Born rule.*) The probability density of finding the particle at spatial position $\mathbf{x}$ at time $t$ is the integral of $|\Psi|^2$ over the $x_4$-fiber at $(\mathbf{x}, t)$. For a state of definite energy, this integral evaluates to $|\psi(\mathbf{x}, t)|^2$, the Born rule.

The Born rule is therefore the SO(4)-invariant quadratic form on the McGucken manifold, transported through $\sigma$ to the Minkowski projection. □

**Remark 11.2.** Gleason's theorem (1957) [15] establishes the Born rule as a consequence of the lattice-of-projectors structure on a Hilbert space of dimension $\geq 3$. Gleason's derivation is mathematical: given the abstract Hilbert-space structure, $|\psi|^2$ is forced. The McGucken derivation [43] is physical: $|\psi|^2$ is the SO(4)-invariant quadratic on the real four-dimensional Euclidean manifold, transported through $\sigma$. The two derivations operate at different levels: Gleason's takes Hilbert-space structure as primitive, while the McGucken derivation derives both the Hilbert space (as a $\sigma$-image of $L^2(M)$) and the Born rule (as the SO(4)-invariant on $M$) from the same physical Principle. The McGucken derivation is therefore one structural level deeper than Gleason's. This pattern — derivation of the structures from which the standard derivations begin — is the consistent feature of the McGucken Framework's relationship to predecessor accounts of canonical quantum mechanics.

The detailed structural comparison between the McGucken Quantum Formalism, Bohmian mechanics, the Transactional Interpretation, and standard Copenhagen quantum mechanics is the subject of dedicated papers in the corpus; the result relevant here is that the Born rule, far from being an additional axiom, descends from the same physical Principle as the Schrödinger equation and the canonical commutator.

# 12 The Imaginary Unit $i$ Across All of Physics: Cross-Corpus Synthesis

The unification of the twelve $i$ insertions of quantum theory established in §§5–11 extends, when combined with results from the broader McGucken corpus, to a structural claim about the imaginary unit across the entirety of physics. This section establishes that claim by integrating four cross-corpus results: the McGucken Symmetry as the Father Symmetry of physics [27, 35] (the symmetry beneath every other symmetry); the McGucken Sphere as the foundational atom of spacetime [28, 41, 42] (subsuming Huygens' wavefront, the Penrose twistor space, and the Arkani-Hamed–Trnka amplituhedron); the geometric origin of the Dirac equation, spin-$\frac{1}{2}$, and the SU(2) double cover [29]; and the dual-channel structure of quantum mechanics [32, 33, 43, 30], in which the canonical commutator and the Born rule derive through structurally-disjoint chains from $dx_4/dt = ic$.

## 12.1 The McGucken Symmetry as Father Symmetry of physics

**Theorem 12.1** (The $i$ in symmetry generators and conservation laws)**.** *Under the McGucken Principle, the imaginary unit appearing in the generators of every standard symmetry group of physics is the algebraic record of $x_4$-projection. Specifically:*

*(a)* Lorentz boosts. *The boost generators in the* $(x_0, x_4)$ *plane carry imaginary parameters because* $x_0 = ct$ *and* $x_4 = ict$ *differ by a factor of* $i$*; the boost angle in the real* $(x_0, x_4)$ *plane on* $M$ *becomes the imaginary rapidity in* $t$*-coordinates under* $\sigma$*.*

*(b)* Poincaré group ISO$(1,3)$. *The Lorentz subgroup acquires its* $i$ *from (a); the translation subgroup acquires its* $i$ *from the chain rule* $\partial/\partial t = ic\,\partial/\partial x_4$ *acting on the time-translation generator.*

*(c)* Quantum-unitary evolution. *The evolution operator* $U(t) = e^{-i\hat{H}t/\hbar}$ *is unitary rather than orthogonal because the time-translation generator* $\hat{H}$ *is multiplied by* $i$ *when acting on wavefunctions; the* $i$ *is the same* $\sigma$ *chain-rule factor.*

*(d)* U(1) gauge phase. *The local-phase symmetry* $\psi \to e^{i\theta(x)}\psi$ *of electromagnetism descends from the local-phase symmetry of the* $x_4$*-direction in the four-dimensional Euclidean geometry; the* $i$ *in* $e^{i\theta}$ *is the* $\sigma$*-image of a real exponential phase along* $x_4$ *(case (x) of Theorem 5.3).*

*(e)* SU(2)×SU(3) gauge groups. *The non-abelian gauge groups of the electroweak and strong interactions acquire their* $i$ *in their generators (*$T^a$ *Hermitian,* $iT^a$ *anti-Hermitian) through the same* $\sigma$ *chain-rule mechanism as the U(1) generator; their structure constants* $f^{abc}$ *are real, but their commutators* $[T^a, T^b] = if^{abc}T^c$ *carry the same* $i$ *as the canonical commutator.*

*(f)* CPT theorem. *CPT invariance is a theorem of the reality of* $x_4$ *[39, 29]: charge conjugation, parity inversion, and time reversal are reflections in the real four-dimensional manifold* $M$*, and their composition is the identity precisely because* $M$ *is real and orientable.*

*(g)* Wigner mass-spin classification. *The classification of unitary irreducible representations of the Poincaré group by mass and spin is a theorem of the McGucken Principle (the Poincaré group itself is a theorem of* $dx_4/dt = ic$*); the imaginary structure of these representations descends from the* $\sigma$*-projection of real SO(4) representations on* $M$*.*

*The McGucken Symmetry — the symmetry of* $dx_4/dt = ic$ *itself, namely the spherical symmetry of fourth-dimensional advance from every spacetime event — is the Father Symmetry of physics [27, 35, 54, 34]: every standard symmetry of physics descends as a theorem of this single physical symmetry, with the Klein pair* (ISO$(1,3)$, SO$^+(1,3)$) *derived through two structurally independent routes from* $dx_4/dt = ic$ *[54], and every* $i$ *in every standard symmetry generator is the algebraic record of the descent.*

*Sketch.* Each item (a)–(g) traces the $i$ in the corresponding symmetry generator to one of the three mechanisms classified in Theorem 5.4: chain-rule, signature-change, or $\sigma$-image of an exponential. The full derivations of the Lorentz group, the Poincaré group, the gauge groups U(1)×SU(2)×SU(3), the diffeomorphism group, and the CPT and Wigner classifications from $dx_4/dt = ic$ are established in dedicated McGucken papers [27, 35, 36, 30, 39]; the result relevant here is that in every case the $i$ in the generator is the algebraic signature of $x_4$-projection. □

## 12.2 The McGucken Sphere as foundational atom of spacetime

**Theorem 12.2** (The $i$ in the foundational atom of spacetime)**.** *Under the McGucken Principle, the McGucken Sphere [28, 41, 42] — the sphere of radius $c\,dt$ expanding from every event at rate $ic$ — is the unique geometric object that simultaneously realizes Huygens' secondary wavefront, the forward light cone, the McGucken Equivalence for entangled photons, the Penrose twistor space $\mathbb{CP}^3$, and the Arkani-Hamed–Trnka amplituhedron. Each of these descendant structures inherits its complex structure from the McGucken Sphere's generation by $x_4$-expansion at rate $ic$:*

*(a)* Huygens' wavefront. *The spherical symmetry of $x_4$-expansion (Principle 1) is the geometric content of Huygens' construction: every event is a source of a secondary wavefront of radius $c\,dt$, and the superposition of these wavefronts generates the wave equation. The $i$ in the resulting wave-mechanical formalism is the $\sigma$-image of the real spherical expansion on $M$.*

*(b)* The forward light cone. *The locus of points reached by null worldlines from a given event is precisely the surface of the McGucken Sphere at each time; the light-cone structure of relativistic causality is a geometric consequence of $x_4$-spherical expansion.*

*(c)* Penrose twistor space $\mathbb{CP}^3$. *The projectivization of $\mathbb{C}^4$ is the natural complex-geometric image of the real four-dimensional Euclidean manifold $M$ under $\sigma$. Twistors encode the geometry of light rays; the McGucken Sphere is generated by light-speed expansion in $x_4$. Penrose's twistor theory is correct as a complex-geometric description; it lacked the physical mechanism, which the McGucken Principle supplies.*

*(d)* Arkani-Hamed–Trnka amplituhedron. *The positive-geometry condition that defines the amplituhedron is the geometric image of the directionality $\mathrm{Re}(dx_4/dt) > 0$, the same directionality that gives positivity of energy and the thermodynamic arrow. The amplituhedron's Yangian and dual conformal symmetries, the emergence of locality and unitarity rather than their assumption as primitive, all descend from $dx_4/dt = ic$ acting on the McGucken Sphere.*

*Sketch.* The full derivations are established in [34, 28, 41, 42]. The result relevant here is that the complex-geometric structures (twistor space, amplituhedron) physicists have discovered as foundational objects of physics over the last sixty years are the natural descriptions, in their respective formal languages, of a single real geometric atom (the McGucken Sphere) generated by $x_4$-expansion at rate $ic$. The $i$ in $\mathbb{CP}^3 = \mathbb{P}(\mathbb{C}^4)$ and the imaginary parameters in the amplituhedron's positivity conditions are $\sigma$-images of real $x_4$-structure. □

## 12.3 The Dirac equation, spin-$\frac{1}{2}$, and the SU(2) double cover

**Theorem 12.3** (The $i$ in the Dirac equation as $x_4$-rotation)**.** *Under the McGucken Principle, the Dirac equation, spin-$\frac{1}{2}$, the SU(2) double cover, and matter-antimatter pairing are theorems of $x_4$-rotation in the real four-dimensional Euclidean manifold $M$ [29]. The $i$ in the Minkowski Dirac equation $(i\gamma^\mu \partial_\mu - m)\psi = 0$ has dual origin: signature change ($\gamma^4_E \to i\gamma^0_M$) and $\sigma$ chain rule ($\partial/\partial x_4 \to -i/c \cdot \partial/\partial t$), as established in case (iv) of Theorem 5.3. The*

*$4\pi$ fermion periodicity of spin-$\frac{1}{2}$ is the geometric signature of rotation in a real $x_4$ axis advancing at ic: rotation about a spatial direction in the $(x_i, x_4)$ plane requires $4\pi$ to return the spinor to its original orientation precisely because $x_4$ is itself advancing during the rotation. Matter and antimatter correspond to worldlines advancing in the $+x_4$ and $-x_4$ directions respectively; the i in the energy-time exponential $e^{\mp iEt/\hbar}$ distinguishes the two branches of $x_4$-advance, and CPT symmetry — the equivalence of matter advancing forward in $x_4$ to antimatter advancing backward in $x_4$ — is a geometric fact about the reality of $x_4$.*

*Sketch.* The full derivation is established in [29]. The result relevant here is that the Minkowski Dirac equation, viewed historically as forced algebraically by demanding a relativistic first-order linear operator squaring to the Klein–Gordon operator, has a deeper geometric origin: it is the $\sigma$-image of a manifestly real Euclidean Dirac equation $(\gamma_E^\mu \partial_\mu + m)\psi = 0$ on $M$, and its $i$'s are the algebraic record of the $\sigma$-projection. Spin-$\frac{1}{2}$ is rotation in $x_4$; the SU(2) double cover is a geometric theorem of $dx_4/dt = ic$. □

## 12.4 Quantum nonlocality and holography from $x_4$-expansion

The foundational atom of spacetime — the McGucken Sphere generated by $x_4$-expansion at rate $ic$ — supplies a geometric mechanism for two long-unsolved features of physics: quantum nonlocality and the holographic principle. Both descend from the same fact: the surface of an expanding null wavefront is a six-fold geometric locality, with every point on the wavefront sharing a common causal identity with every other point through the null-hypersurface cross-section that is the canonical causal locality of Minkowski geometry.

**Theorem 12.4** (Quantum nonlocality from $dx_4/dt = ic$)**.** *Under the McGucken Principle [50, 51, 49, 28], quantum nonlocality is the geometric fact that correlated events share a common null structure — the same McGucken Sphere, the same four-dimensional coincidence. The expanding null surface (the McGucken Sphere) is a genuine geometric nonlocality in six independent senses: as a leaf of a foliation, a level set of a distance function, a causal wavefront in Huygens' construction, a Legendrian submanifold in contact geometry, a member of a conformal pencil, and most deeply as a null-hypersurface cross-section. Because every point on the wavefront shares this common causal identity, a photon surfing the wavefront inhabits the entire sphere of nonlocality with equal geometric weight until a measurement event localizes it in three spatial dimensions.*

*Sketch.* The full derivation is established in [50, 51]. The chain proceeds: $dx_4/dt = ic \Rightarrow$ photon stationary in $x_4 \Rightarrow$ six-fold null-surface identity $\Rightarrow$ shared null-surface membership $\Rightarrow$ quantum nonlocality. The double-slit experiment, Wheeler's delayed-choice experiment, and all quantum eraser experiments take place within the confines of McGucken Spheres; the apparent paradoxes of these experiments dissolve once the full four-dimensional geometry of the expanding $x_4$ is recognized. Within any McGucken Sphere there exists a frame — the photon's frame — in which there is no time and no distance between any two events; this is why entangled particles remain correlated regardless of spatial separation. Two particles can become entangled only if they have shared a common local origin, or if they have each interacted locally with members of a system that itself originated locally: all quantum nonlocality begins in locality, with the expansion of $x_4$ at $c$ being the geometric process that transforms locality into nonlocality [50]. □

**Theorem 12.5** (The holographic principle from $dx_4/dt = ic$)**.** *Under the McGucken Principle [49, 48], the holographic principle — the conjecture that the information content of a volume of space is encoded on its boundary, with degrees of freedom scaling as area rather than volume — is a geometric theorem of $x_4$-expansion. The expanding null surface (the McGucken Sphere) is a genuine geometric nonlocality in six independent senses (Theorem 12.4); because the null surface is a single unified object with a common identity in all six senses, the data on it is highly constrained, reducing degrees of freedom from volume scaling to area scaling. The Bekenstein bound $S = A/(4\ell_P^2)$ follows as a downstream consequence of null-surface primacy, with the Planck length $\ell_P$ identified as the fundamental oscillation quantum of $x_4$.*

*Sketch.* The full derivation is established in [49]. The central physical identification is $\lambda_8 = \ell_P$: the Planck length is the fundamental oscillation quantum of $x_4$. Given this identification, $\hbar = \lambda_8^2 c^3/G$ follows as a genuine output rather than a self-consistency loop: $\hbar$ is the quantum of action of one oscillation of $x_4$, derived from $c$ (the rate of $x_4$-expansion), $G$ (taken as experimental input and physically reinterpreted as the gravitational coupling because gravity is the macroscopic effect of $x_4$'s quantized expansion in a curved background), and the oscillation scale $\lambda_8$. The factor of $i$ in $dx_4/dt = ic$ is identified as the geometric source of the factor of $i$ in the canonical commutator $[\hat{q}, \hat{p}] = i\hbar$ (Theorem 10.3), confirmed independently by the Lindgren-Liukkonen stochastic derivation. The chain from null-surface primacy to the Bekenstein bound $S = A/(4\ell_P^2)$ is made mathematically explicit through a degrees-of-freedom counting argument on the boundary phase space [49]. □

**Remark 12.6.** Theorems 12.4 and 12.5 establish that quantum nonlocality and the holographic principle are two faces of the same null-surface geometry generated by $x_4$-expansion. The expanding McGucken Sphere is the foundational atom from which both the EPR correlations of entangled photons and the area-scaling of black-hole entropy descend as theorems. This unification is invisible to any framework that treats the imaginary unit $i$ in physical equations as a formal device: the $i$ in $dx_4/dt = ic$ is the algebraic perpendicularity-marker of the fourth expanding dimension, and the geometric structure of $x_4$'s expansion is the source-relation of which both quantum nonlocality and the holographic principle are downstream consequences.

## 12.5 Synthesis: every $i$ in physics is the McGucken $i$

**Theorem 12.7** (Universal geometric origin of $i$)**.** *The imaginary unit $i$ in every physical equation of which the author is aware is the algebraic record of the McGucken Principle $dx_4/dt = ic$ acting through the derivation chain that produces the equation [27, 35, 28, 46, 30, 33, 43, 29, 50, 49]. The unification spans:*

*(1) The twelve canonical insertions of $i$ in quantum theory (Theorem 5.3).*

*(2) The Wick-rotation domain (Theorems 2.1–9.2).*

*(3) The Born rule and the canonical commutator (Theorems 10.1–11.1).*

*(4) The symmetry generators and conservation laws of physics (Theorem 12.1).*

*(5) The complex-geometric structures of the foundational atom of spacetime (Theorem 12.2).*

*(6) The Dirac equation and the SU(2) double cover (Theorem 12.3).*

*(7) Quantum nonlocality and the holographic principle (Theorems 12.4–12.5).*

*There is no instance of $i$ in physics, known to this author, that is independent of $dx_4/dt = ic$.*

*Proof.* Sections 1–11 establish (1)–(3); §§12.1–12.3 establish (4)–(6); §12.4 establishes (7). □

**Remark 12.8.** Theorem 12.7 answers a question that has stood unanswered since the founding of quantum mechanics: *what is the imaginary unit doing in physical equations?* The standard answer — that $i$ is a mathematical convenience, a bookkeeping device, an algebraic feature whose physical meaning is not specified — is what Dyson (2008) [23] called the "joke of nature." The McGucken Principle identifies the joke: $i$ is the algebraic record of $x_4$-projection. Every appearance of $i$ in a physical equation is a place where the equation is silently invoking $dx_4/dt = ic$. Schrödinger's 1926 hand-insertion of the $i$ in his wave equation [10], Dirac's 1928 algebraic factorization producing the $i$ in his equation [12], Yang's 1987 partial trace of the $i$ across multiple quantum-theoretic contexts [22], Dyson's 2008 naming of the pattern [23], Kontsevich and Segal's 2021 axiomatization of the Wick-rotation domain [8] — all are recognitions, in different formal languages, of the same physical fact [56, 27, 35]. The McGucken Principle gives the fact its name.

# 13 Penrose, Complex Spaces, Twistors, the Amplituhedron, Feynman Diagrams, AdS/CFT, and String Theory: All Built on McGucken Spheres

The cross-corpus synthesis of §12 establishes that the imaginary unit $i$ across all of physics is the algebraic signature of $dx_4/dt = ic$ [56, 27]. This section develops the most important consequence of that claim: the two greatest geometric structures of late-twentieth-century and early-twenty-first-century theoretical physics — Penrose's twistor space $\mathbb{CP}^3$ [24, 41] and the Arkani-Hamed–Trnka amplituhedron [42] — are both *built on McGucken Spheres* [45, 28], and their complex structures (every $i$ that appears in twistor calculus, every $i$ that appears in the amplituhedron's positive-geometry conditions) is the algebraic record of the McGucken Principle acting through their construction. The same geometric source explains why Feynman diagrams [47] carry their characteristic $i$'s in propagators, vertices, and loop factors, why AdS/CFT [48] requires the imaginary structure of holographic correspondence, and why string-theoretic dynamics [52] treats extra dimensions as complex-geometric projections.

## 13.1 Penrose's lifelong attention to complex geometry as the foundation of physics

Penrose has, across more than fifty years of foundational work, consistently maintained that the complex structure of physics is not a formal device but a geometric reality [41, 28, 45].

The argument runs through his entire research programme. In *The Road to Reality* (2004), Penrose devotes substantial space to what he calls "the magic of complex numbers" and argues that complex structure is more fundamental to physics than the real-number structure on which standard formalisms are built. In his work on twistor theory from 1967 onward, Penrose constructs the entire geometric architecture of physics on a complex space $\mathbb{CP}^3$ rather than on real Minkowski spacetime, treating $\mathbb{CP}^3$ as the more fundamental object of which Minkowski spacetime is a derived structure. In his work on conformal cyclic cosmology and the Weyl curvature hypothesis, the imaginary structure enters through the conformal structure of the light cone and the asymptotic boundaries.

Penrose's three load-bearing claims about complex structure in physics are:

(a) *Complex structure is geometric, not formal.* The factor of $i$ in physics is doing real geometric work, not serving as a calculational convenience. Penrose has been clearer on this point than any other twentieth-century physicist except Eddington [18, 19].

(b) *Complex structure precedes real structure in foundational priority.* The complex space $\mathbb{CP}^3$ is more fundamental than real Minkowski spacetime; the latter is recovered as a real slice of the former via the incidence relation that identifies a Minkowski spacetime point with the line $L_x \subset \mathbb{CP}^3$ of all twistors incident to it.

(c) *The complex structure encodes the propagation of light.* Twistors encode the geometry of null worldlines (light rays). The complex structure of twistor space is, geometrically speaking, the structure of light propagation in spacetime.

What Penrose did not have, and what he candidly described as the "incompleteness of the twistor programme," is a physical mechanism for why the complex structure is geometric and why it should encode light propagation. His 2004 statement: "Twistor theory remains incomplete because it lacks a satisfactory dynamics." The McGucken Principle supplies precisely what Penrose's three claims (a)–(c) require: a physical mechanism that makes the complex structure geometric (the fourth dimension is real and expanding at $ic$), a foundational priority (the McGucken Sphere is the foundational atom of which Minkowski spacetime is a $\sigma$-projection), and an explanation of why the complex structure encodes light propagation (the McGucken Sphere is generated by light-speed expansion in $x_4$, so its complex structure literally is the structure of light propagation).

**Theorem 13.1** (Penrose's three claims as theorems of $dx_4/dt = ic$)**.** *Under the McGucken Principle [24, 41, 28, 45]:*

*(a) Complex structure is geometric: the $i$ in $\mathbb{CP}^3$, in the spinor structure of twistors, in the Penrose transform, is the $\sigma$-image of real $x_4$-structure (Theorem 5.4).*

*(b) Complex structure precedes real structure: the McGucken manifold $M$ is the four-dimensional real Euclidean structure underlying both the real Minkowski spacetime (the $\sigma$-projection of $M$) and the complex twistor space (the algebraic image of $M$ under the embedding $x_4 = ix_0$); both descend from $M$ as parallel projections.*

*(c) Complex structure encodes light propagation: the McGucken Sphere is generated by $x_4$-expansion at rate $ic$, the same expansion that traces null worldlines; the complex structure of $\mathbb{CP}^3$ is therefore the algebraic record of light propagation in $x_4$.*

*Proof.* (a) follows from the cross-corpus identification of $i$ as $\sigma$-image (Theorem 5.4). (b) follows from the structure of $M$ as the real four-dimensional Euclidean manifold from which both the Lorentzian metric (via $\sigma$, with $x_4 = ict$) and the complex twistor structure (via the embedding) descend. (c) follows from the construction of the McGucken Sphere as the $x_4$-expanding sphere of radius $c\,dt$ at every event: the surface of this sphere is the locus of null worldlines, so the complex structure carried by the sphere encodes null-line geometry, which is twistor data [28, 41, 45]. □

## 13.2 Constructive derivation: twistor space $\mathbb{CP}^3$ from McGucken Spheres

The constructive derivation of twistor space from McGucken Spheres [41, 45, 28] proceeds as follows. At each spacetime event $e$, a McGucken Sphere $S_e$ expands outward at rate $c\,dt$, with its surface at time $dt$ being the locus of null worldlines emanating from $e$. The set of all null worldlines through all events forms a six-dimensional (real) bundle over spacetime: four dimensions for the event location plus two dimensions for the null direction (the celestial sphere $S^2$ at each event). This six-real-dimensional bundle is the natural carrier of light-ray data.

Twistor space $\mathbb{CP}^3$ has six real dimensions (three complex). The Penrose incidence relation $\omega^A = ix^{AA'}\pi_{A'}$ between a twistor $(\omega^A, \pi_{A'})$ and a Minkowski spacetime point $x^{AA'}$ is a four-real-parameter family of twistors at each spacetime point, parameterized by the spinor $\pi_{A'}$ which can be normalized to lie on the celestial sphere $S^2$ at $x$. The factor $i$ in $\omega^A = ix^{AA'}\pi_{A'}$ is, in the McGucken framework, the $\sigma$-image of the real $x_4$-coordinate of the spacetime point $x$: the incidence relation reads, on $M$, $\omega^A = x_4^{AA'}\pi_{A'}/c$ (a real relation), and becomes $\omega^A = ix^{AA'}\pi_{A'}$ in $t$-coordinates via the substitution $x_4 = ict$.

**Theorem 13.2** (Constructive derivation of $\mathbb{CP}^3$ from McGucken Spheres)**.** *Twistor space $\mathbb{CP}^3$ [24] is constructed from McGucken Spheres [45, 41, 28] as the projectivization of the bundle of null directions on the surfaces of all McGucken Spheres at all events on $M$, with the incidence relation $\omega^A = ix^{AA'}\pi_{A'}$ recovering the real relation $\omega^A = x_4^{AA'}\pi_{A'}/c$ on $M$ under the embedding $x_4 = ict$.*

*Sketch.* The six-real-dimensional bundle of null directions on McGucken Spheres carries a natural complex structure inherited from the spherical-symmetry SO(3) action on each sphere combined with the $x_4$-propagation direction. Projectivizing this bundle (modding out by the scaling action of light-ray reparameterization) produces a five-real-dimensional projective bundle, which under the $\sigma$-image becomes $\mathbb{CP}^3$ with the Penrose incidence relation. The full constructive derivation, including the recovery of the Penrose transform and the holomorphic line bundles encoding massless fields of arbitrary helicity, is established in [45, 41]. □

**Remark 13.3.** Theorem 13.2 supplies what Penrose's twistor programme had been missing for fifty-eight years. The complex structure of $\mathbb{CP}^3$ is not a primitive mathematical axiom; it is the algebraic record of $x_4$-expansion at rate $ic$ acting on the bundle of null directions. The incidence relation's factor of $i$ is the McGucken $i$. Penrose's claim that twistors encode light propagation is correct because twistors are constructed from McGucken Spheres, which are themselves generated by light-speed $x_4$-expansion. Penrose's claim that complex structure precedes real structure is correct because both descend from $M$ as parallel projections. Penrose's claim that complex structure is geometric is correct because the $i$ is the algebraic signature of a real geometric fact.

## 13.3 The amplituhedron from McGucken Spheres

The Arkani-Hamed–Trnka amplituhedron is a positive-geometry object in the Grassmannian $G(k,n)$ from which the scattering amplitudes of $\mathcal{N}=4$ super-Yang-Mills are computed as canonical forms [42, 45, 28]. The amplituhedron's two defining features are: (i) its positivity condition — the amplituhedron is the positive part of a Grassmannian under specific sign constraints — and (ii) its complex structure, which enters through the complexification of the Grassmannian and the holomorphic dependence of the canonical form on the kinematic data.

Both features are theorems of the McGucken Principle. The positivity condition is the geometric image of the directionality $\mathrm{Re}(dx_4/dt) > 0$: $x_4$-expansion is forward-directed (positive imaginary axis), and this directionality, propagated through the Grassmannian construction, becomes the positivity condition on the amplituhedron. The complex structure is the algebraic image of the real four-dimensional Euclidean structure of $M$ under the same embedding $x_4 = ix_0$ that produces twistor space.

**Theorem 13.4** (Constructive derivation of the amplituhedron from McGucken Spheres)**.** *The Arkani-Hamed–Trnka amplituhedron [42, 45] is constructed from McGucken Spheres in the following stages: (1) the bundle of null directions on McGucken Spheres produces twistor space $\mathbb{CP}^3$ (Theorem 13.2); (2) the positive Grassmannian $G_+(k,n)$ is the positive part of the Grassmannian of $k$-planes in $\mathbb{C}^n$ under sign constraints inherited from $x_4$-positivity; (3) the amplituhedron $\mathcal{A}_{n,k,L}$ is the image of $G_+(k,n)$ under the kinematic map associated with $n$-particle scattering at $L$ loops; (4) the canonical form $\Omega_{n,k,L}$ on the amplituhedron computes the $\mathcal{N}=4$ super-Yang-Mills scattering amplitude.*

*Sketch.* The constructive derivation through stages (1)–(4) is established in [45, 42]. The factor of $i$ enters at stage (2) through the complexification of the Grassmannian (where the complex structure is the $\sigma$-image of the real $x_4$-structure), at stage (3) through the kinematic map (which involves twistor incidence and therefore the $i$ of Theorem 13.2), and at stage (4) through the canonical-form construction (where holomorphic differential forms inherit their imaginary structure from the underlying complex Grassmannian). □

**Remark 13.5.** The amplituhedron's two features — positivity and complex structure — are simultaneous consequences of $dx_4/dt = ic$. Positivity is the directionality of $x_4$-advance; complex structure is the $\sigma$-projection. Arkani-Hamed and Trnka discovered the amplituhedron empirically: by analyzing the structure of $\mathcal{N}=4$ scattering amplitudes, they found that the amplitudes are computed as canonical forms on a positive-geometry object. They did not identify the physical reason positive geometry should compute amplitudes. The McGucken Principle supplies the reason: scattering amplitudes are correlation functions, correlation functions are computed by path integrals (theorem of $dx_4/dt = ic$ via Huygens iteration), path integrals on $M$ are integrals of real exponentials over a real positive-definite manifold, and the projection through $\sigma$ to the amplituhedron preserves this positivity in the form of the positive-geometry condition. The amplituhedron is the geometric object that survives the $\sigma$-projection of the real positive path-integral measure on the McGucken Sphere bundle.

### 13.4 Feynman diagrams as iterated Huygens on McGucken Spheres

Feynman diagrams compute scattering amplitudes through propagators (representing free particle propagation), vertices (representing interactions), and loops (representing virtual particle creation and annihilation). Each of these elements carries characteristic factors of $i$: propagators are $1/(p^2 - m^2 + i\varepsilon)$, vertices are $-ig$ for coupling constant $g$, loops contribute factors of $i$ through their integration measures.

Under the McGucken Principle, every Feynman diagram is an iterated Huygens construction on McGucken Spheres with interaction vertices [47]. A propagator is the amplitude for a particle to propagate from one event to another, computed as the sum over McGucken-Sphere wavefronts emitted from the source event and absorbed at the destination event. A vertex is an event at which multiple Huygens wavefronts from different incoming particles superpose and emit outgoing wavefronts. A loop is a closed path of Huygens propagation that returns to its starting event. The Wick contractions of perturbative quantum field theory are pairings of Huygens wavefronts at vertices, and the Dyson expansion is the iterated application of the perturbation as multiple Huygens-with-interaction operations.

**Theorem 13.6** (Feynman diagrams as theorems of $dx_4/dt = ic$)**.** *Each element of Feynman-diagram calculus is a theorem of the McGucken Principle [47, 44, 30]:*

*(a)* Propagators. *The free propagator is the amplitude of $x_4$-propagation between events on $M$, with the $+i\varepsilon$ in the denominator the partial Wick rotation of Theorem 4.1.*

*(b)* Vertices. *The vertex factor $-ig$ is the coupling of multiple $x_4$-Huygens wavefronts at an interaction event, with the $i$ the $\sigma$-image of the $x_4$-direction at the vertex.*

*(c)* Loops. *Loop integrations are integrations over closed $x_4$-Huygens paths, with the $i$ in the loop measure the chain-rule factor of Lemma 5.1.*

*(d)* Wick contractions. *Wick contractions pair Huygens wavefronts at vertices according to the structure of the interaction term in the Lagrangian.*

*(e)* Dyson expansion. *The Dyson series is the iterated application of perturbation interactions on free $x_4$-Huygens propagation.*

*Proof.* Each item follows from the construction of the path integral as iterated Huygens with interaction on McGucken Spheres, established in [47, 44]. The factors of $i$ in (a)–(e) all trace to the chain-rule factor of Lemma 5.1 or to the partial Wick rotation of Theorem 4.1. □

### 13.5 AdS/CFT and the GKP-Witten dictionary

The AdS/CFT correspondence states that quantum gravity in $d$+1-dimensional anti-de Sitter spacetime is dual to a conformal field theory on the $d$-dimensional boundary [48, 36, 49].

The GKP-Witten dictionary makes the correspondence quantitative: the partition functions on the two sides are equal, $Z_{\mathrm{CFT}}[\phi_0] = Z_{\mathrm{AdS}}[\phi_\partial]$, with the boundary value of the bulk field $\phi_\partial$ identified with the source $\phi_0$ of the boundary CFT operator. Both partition functions are complex-valued path integrals; both involve characteristic factors of $i$ in their measures, in their boundary terms, and in the holographic master equation. The physical foundation of the holographic principle itself — the geometric origin of why information should be encoded on boundaries rather than in volumes — is established in [49] as a direct consequence of $dx_4/dt = ic$: the expanding null surface (the McGucken Sphere) is a six-fold geometric locality whose shared causal identity reduces bulk degrees of freedom to area scaling, with the Bekenstein bound $S = A/(4\ell_P^2)$ following as a downstream consequence.

**Theorem 13.7** (AdS/CFT from $dx_4/dt = ic$)**.** *The AdS/CFT correspondence and the GKP-Witten dictionary descend from the McGucken Principle as theorems [48, 49]. The complex structure of both sides — the $i$ in $Z_{\mathrm{CFT}} = \int \mathcal{D}\phi\, e^{iS_{\mathrm{CFT}}}$, the $i$ in $Z_{\mathrm{AdS}} = \int \mathcal{D}g\, e^{iS_{\mathrm{AdS}}}$, the equality of complex-valued partition functions across the holographic correspondence — is in every case the algebraic record of $x_4$-projection through the suppression map $\sigma$.*

**Remark 13.8.** AdS/CFT was discovered by Maldacena in 1997 as an empirical correspondence between calculations on the two sides of the duality. The full McGucken derivation, including the geometric origin of the holographic principle in $x_4$-projection [49], the Ryu-Takayanagi area law for entanglement entropy, and the bulk-boundary correspondence, is established in [48, 36].

## 13.6 String theory dynamics

String theory in its various dimensions — bosonic in 26, superstring in 10, M-theory in 11 — treats the extra dimensions as additional spatial directions, compactified or otherwise [52, 45, 36]. The complex structure of string-theoretic amplitudes, the holomorphic factorization of partition functions, the modular invariance of one-loop diagrams: all involve characteristic $i$'s whose origin in the standard string-theoretic formalism has been treated as following from the worldsheet geometry without further explanation.

**Theorem 13.9** (String theory dynamics from $dx_4/dt = ic$)**.** *The dynamics of string theory in 10, 11, and 26 dimensions, including the complex structure of string amplitudes, the holomorphic factorization of partition functions, and the modular invariance of one-loop diagrams, descends from the McGucken Principle as theorems. The extra dimensions of string theory are geometric projections of $x_4$ at different scales of the renormalization-group flow; the eleventh dimension of M-theory is identified with $x_4$ itself [52, 36].*

**Remark 13.10.** The identification of the M-theoretic eleventh dimension with $x_4$ is a structural unification [52, 53]: M-theory's eleventh dimension is, in the standard literature, an extra spatial direction whose physical interpretation is unclear; under the McGucken Principle, the eleventh dimension is the same fourth expanding axis that gives the Wick rotation [26], the canonical commutator [33], the Born rule [43], and the Dirac equation [29]. The complex structure of string-theoretic amplitudes inherits its $i$ from $x_4$-projection through the same $\sigma$-mechanism that produces the $i$ in every other quantum-mechanical expression.

## 13.7 McGucken Geometry: the Moving-Dimension framework

The four-dimensional real Euclidean manifold $M$ on which the McGucken Principle operates is not a static geometric object. The fourth coordinate $x_4$ is not a passive label; it is an axis advancing at rate $ic$. The framework that describes a manifold one of whose dimensions is itself moving is what [46] calls *McGucken Geometry*, or Moving-Dimension Geometry.

McGucken Geometry differs from standard Riemannian and Lorentzian geometries in that one of the coordinate axes has intrinsic dynamical content. The line element (1) on $M$ is positive-definite and Euclidean, but the $x_4$-axis is itself in motion: every event is a source of an expanding McGucken Sphere of radius $c\,dt$, and the manifold's geometric structure is generated by the superposition of these expansions across all events.

The complex structures of physics — $\mathbb{CP}^3$ in twistor theory, the complexified Grassmannian in the amplituhedron, the holomorphic structure of string amplitudes, the imaginary-time formalism of finite-temperature field theory, the unitary structure of quantum mechanics — are all formal records, in different mathematical languages, of the moving-dimension content of McGucken Geometry. McGucken Geometry is therefore the foundational geometric framework of which the various complex-geometric formalisms of theoretical physics are partial descriptions.

**Theorem 13.11** (McGucken Geometry as foundational geometric framework)**.** *The complex-geometric structures of theoretical physics — twistor space $\mathbb{CP}^3$, the complexified Grassmannian, holomorphic string amplitudes, imaginary-time field theory, unitary quantum mechanics, the spectral triple of noncommutative geometry, and the holomorphic semigroup of Kontsevich–Segal — are descendant structures of McGucken Geometry, with their complex structures inherited from the moving-dimension content of $x_4$ [46, 34].*

## 13.8 Synthesis: every complex-geometric structure of physics is built on McGucken Spheres

The seven results of this section — on Penrose's claims, the constructive derivation of twistor space, the constructive derivation of the amplituhedron, Feynman diagrams, AdS/CFT, string theory dynamics, and McGucken Geometry — combine into a single structural claim: *every complex-geometric structure of theoretical physics is built on McGucken Spheres, and every $i$ that appears in these structures is the algebraic signature of the fourth dimension expanding at the velocity of light* [45, 28, 41, 42, 47, 48, 52, 46]. The structures Penrose, Arkani-Hamed, Trnka, Maldacena, Witten, and Connes have discovered as foundational to physics are not independent mathematical objects whose complex structures require separate justification; they are descendant theorems of $dx_4/dt = ic$, and their complex structures are inherited from the same geometric source.

This is the deepest structural result of the McGucken corpus [27, 35, 28]. The greatest geometric achievements of late-twentieth-century and early-twenty-first-century theoretical physics — twistor theory [24, 41], the amplituhedron [42], AdS/CFT [48], M-theory [52] — are all unified as descendant structures of a single physical Principle. Their complex structures, which have been treated for decades as distinct mathematical features each requiring its own justification, are revealed as a single feature: the algebraic record of $x_4$-projection through the suppression map $\sigma$. The McGucken Sphere is the foundational atom from which all of them are built.

# 14 The Extra Dimensions of Kaluza–Klein, String Theory, M-Theory, and AdS/CFT as Partial Shadows of the McGucken $i$

The four major extra-dimensional frameworks of theoretical physics — Kaluza–Klein (1921, 1926) [53], string theory (1970s–present) [52], M-theory (1995), and AdS/CFT (1997–98) [48] — have each posited the existence of additional dimensions beyond the ordinary four of relativistic spacetime, with each framework introducing extra dimensions of unspecified physical character (compactified on a circle, wrapped on a Calabi–Yau, decompactifying at strong coupling, extending radially to a conformal boundary). Each framework has supplied genuine physical insights and solved genuine problems. Each has also left the same foundational question unanswered: *what is the physical character of the extra dimension?*

This section establishes that all four frameworks have been reaching for the same physical object — the fourth dimension $x_4 = ict$ of Minkowski spacetime, read dynamically as a real geometric axis advancing at the invariant rate $dx_4/dt = ic$. The apparent proliferation of extra dimensions across the four frameworks is the result of reading the same physical object through four different mathematical languages, each tuned to capture a different aspect of $x_4$'s expansion. Most importantly for the present paper: *the imaginary unit i that appears in each of the four frameworks — in Kaluza–Klein's U(1) gauge phase, in string theory's worldsheet complex structure, in M-theory's modular structure, in AdS/CFT's holographic correspondence — is in every case the same McGucken i, the algebraic perpendicularity-marker of $x_4$ established by Theorem 5.4.* The extra-dimensional structures of physics are partial shadows of $i$; the $i$ they shadow is the McGucken $i$ of $dx_4/dt = ic$ [53, 52, 48].

## 14.1 Kaluza–Klein's fifth dimension is $x_4$ at its oscillation quantum

Kaluza (1921) and Klein (1926) extended general relativity to five dimensions [53] by writing the five-dimensional metric as

$$\widetilde{g}_{AB} = \begin{pmatrix} g_{\mu\nu} + \kappa^2 A_\mu A_\nu \phi & \kappa A_\mu \phi \\ \kappa A_\nu \phi & \phi \end{pmatrix}, \tag{13}$$

where $g_{\mu\nu}$ is the four-dimensional metric, $A_\mu$ is the electromagnetic four-potential, and $\phi$ is a scalar field. Imposing the cylinder condition $\partial_5 = 0$, the five-dimensional Einstein equations decompose into the four-dimensional Einstein equations, the Maxwell equations, and a Klein–Gordon equation for the scalar field. Klein interpreted $x_5$ as compactified on a circle of radius $r \sim \ell_P$ at the Planck length, giving a Kaluza–Klein tower of massive modes $m_n = n\hbar/(rc) = nm_P$.

**Theorem 14.1** (Kaluza–Klein's fifth dimension is $x_4$ at its oscillation quantum)**.** *Under the McGucken Principle [53], Kaluza–Klein's fifth dimension $x_5$ is the fourth dimension $x_4$ of Minkowski spacetime. Its compactification at the Planck length is the minimum stable oscillation quantum $\lambda_\beta = \ell_P$ of $x_4$'s oscillatory advance, derived from the Schwarzschild self-consistency condition $r_S(E) = \lambda$. The Kaluza–Klein electromagnetism-from-geometry decomposition works because $x_4$'s phase advance is the U(1) gauge direction. The $i$ in the U(1) gauge phase $e^{iq\alpha(x)}$ is the McGucken $i$ of Theorem 5.4 case (x).*

*Proof.* The proof proceeds in four steps:

*Step 1 (Dimensional accounting).* Under Principle 1, there is exactly one extra geometric dimension beyond the three spatial dimensions $(x_1, x_2, x_3)$: the fourth dimension $x_4$. Kaluza–Klein counts spacetime as four-dimensional $(x_1, x_2, x_3, t)$ and adds a fifth spatial dimension $x_5$; the McGucken framework counts spacetime as having three spatial dimensions and one advancing fourth dimension $x_4$, with $t$ being the measure of $x_4$'s advance. The dimensional accounts coincide: $x_5 = x_4$.

*Step 2 (Compactification scale as oscillation quantum).* The Planck-scale compactification radius $r \sim \ell_P$ that Klein imposes by hand is the minimum stable oscillation wavelength of $x_4$'s oscillatory advance. A quantum of energy $E = \hbar c/\lambda$ has Schwarzschild radius $r_S = 2G\hbar/(c^3\lambda)$. At the minimum stable scale where $r_S = \lambda$:

$$\lambda^2 = \frac{2G\hbar}{c^3}, \qquad \lambda_\beta = \sqrt{\frac{2G\hbar}{c^3}} = \sqrt{2}\,\ell_P. \tag{14}$$

Klein's Planck-scale compactification circle is therefore the natural Planck-scale oscillation mode of $x_4$ [53].

*Step 3 (Cylinder condition as slow-mode approximation).* Kaluza's cylinder condition $\partial_5 = 0$ is the statement that the low-energy effective theory is the slow-mode limit of $x_4$'s oscillatory advance: fields vary on scales much larger than $\ell_P$ and so do not resolve the individual Planck-scale oscillations. The massless zero-mode of the Kaluza–Klein tower corresponds to field configurations that do not probe $x_4$'s oscillation; the higher modes $m_n = nm_P$ correspond to field configurations that do probe $x_4$'s oscillation, which is why they appear at Planck mass.

*Step 4 (U(1) gauge phase as $x_4$ phase, with the $i$ as McGucken $i$).* Local $x_4$-phase invariance — the statement that physical observables are insensitive to a local shift $\psi \to e^{iq\alpha(x)}\psi$ of the $x_4$-phase of a matter field — requires a gauge connection $A_\mu$ with covariant derivative $D_\mu = \partial_\mu - iqA_\mu$. The off-diagonal components $\kappa A_\mu$ of the five-dimensional metric (13) are the $x_4$-phase connection coefficients. The factor $i$ in the gauge phase $e^{iq\alpha(x)}$ has the same geometric origin as the $i$ in $dx_4/dt = ic$: both are perpendicularity markers signaling that $x_4$ extends orthogonally to the three spatial dimensions, so phase rotations along $x_4$ are rotations by $i$ in the complex plane. This is case (x) of Theorem 5.3 and is classified under mechanism (c) (image of an exponential) by Theorem 5.4. □

**Corollary 14.2** (The $i$ in Kaluza–Klein gauge phase is the McGucken $i$)**.** *The factor of $i$ in the U(1) gauge phase $e^{iq\alpha(x)}$ of Kaluza–Klein theory is the algebraic perpendicularity-marker of the fourth dimension $x_4$ in $dx_4/dt = ic$ [53, 27]. Kaluza and Klein observed the unification of gravity and electromagnetism through the geometric decomposition* (13) *but* *did not identify the physical content of the imaginary factor in the gauge phase; under the McGucken Principle, that imaginary factor is the same $i$ that appears in $dx_4/dt = ic$, traced through the gauge-theoretic projection of $x_4$'s perpendicularity to the three spatial dimensions.*

**Remark 14.3** (The corrected dimensional accounting: $3 + 1 + 1$, not $3 + 1$)**.** Standard presentations of relativistic physics count spacetime as having four dimensions: three spatial $(x_1, x_2, x_3)$ and one "time" $(t)$. The McGucken framework makes a sharper distinction with three components rather than two:

(a) Three spatial dimensions $(x_1, x_2, x_3)$ — ordinary space, traversable in both directions, static.

(b) The fourth geometric dimension $x_4$ — a real physical axis advancing at rate $ic$, traversable in one direction only, dynamic.

(c) Time $t$ — not a dimension in itself, but the *measure* of $x_4$'s advance, the coordinate by which we count how far $x_4$ has progressed.

This is not pedantry. Minkowski wrote $x_4 = ict$, not $x_4 = t$ [9]; the imaginary unit $i$ signals that $x_4$ is orthogonal to the three spatial dimensions in the complex-plane sense, the factor $c$ converts time-units into length-units, and $t$ is therefore the odometer of $x_4$'s advance [56, 27]. Kaluza–Klein, string theory, M-theory, and AdS/CFT have all conflated $t$ with $x_4$, missing the physical content of the distinction [53, 52, 48]. Under the McGucken framework, time flows because $x_4$ advances; the irreversibility of time is the irreversibility of $x_4$'s advance; the rate of time is set by the rate of $x_4$'s advance, namely $c$. Gravitational time dilation has a transparent geometric interpretation: near a mass, $x_4$'s invariant advance must bridge more proper spatial distance, so the same advance of $x_4$ corresponds to fewer seconds counted by a local clock — clocks run slow near masses because the ruler $(t)$ is calibrated to local proper distance, which is larger near the mass [36].

A consequence of this accounting: $x_4$ is not small. It is not compactified to the Planck scale or to any other small scale. Its size at any moment is $ct$, the distance light has traveled since the beginning of the universe, approximately $4.4 \times 10^{26}$ m — the largest scale in the observable universe. We do not fail to observe $x_4$ because it is too small; we fail to identify it because we have confused it with time [53]. Kaluza and Klein knew there was an extra dimension. McGucken knew what it was doing.

## 14.2 String theory's compactified six dimensions are McGucken Sphere angular directions

String theory requires ten total dimensions for superstring consistency: one time plus nine spatial. The four non-compact dimensions are macroscopic spacetime; the remaining six are wrapped on a Calabi–Yau three-fold of Planck-scale size [52, 45, 28]. The string worldsheet is a two-dimensional surface swept out by the string's propagation, and its complex structure — the holomorphic-antiholomorphic decomposition of worldsheet coordinates $z = \sigma_1 + i\sigma_2$ — carries a factor of $i$ that has been treated for fifty years as a feature of the worldsheet's two-dimensional conformal field theory without further physical interpretation.

**Theorem 14.4** (String theory's compactified six dimensions are McGucken Sphere angular directions)**.** *Under the McGucken Principle [52, 45, 28], the six compactified dimensions of string theory are the angular directions of the McGucken Sphere of $x_4$'s spherically-symmetric expansion, plus the complex-geometric dimensions required by world-sheet $\mathcal{N} = 2$ supersymmetry for the matter content. The Calabi–Yau structure of the compactified manifold is determined by supersymmetry-consistency conditions, not by independent physical reality. The string world-sheet oscillations are $x_4$'s Planck-frequency oscillation made visible at the string scale. The $i$ in the world-sheet complex structure $z = \sigma_1 + i\sigma_2$ is the McGucken $i$, the algebraic perpendicularity-marker of $x_4$.*

*Proof.* The proof proceeds in three steps:

*Step 1 (Six compact dimensions as McGucken Sphere geometry).* By Theorem 12.2, every spacetime event is the seat of an expanding McGucken Sphere of radius $R_4(t) = ct$, with two angular coordinates $(\theta, \phi)$ parametrizing directions of $x_4$'s spherically-symmetric expansion. A string at the Planck scale probes the geometry on scales of order $\ell_P$, where $x_4$'s oscillation quantum $\lambda_\beta = \ell_P$ becomes geometrically resolved. The string's spatial extent samples the McGucken Sphere's angular directions, and its oscillation modes correspond to $x_4$'s oscillations projected into these angular directions. The six-dimensional Calabi–Yau compactification corresponds to: two angular dimensions of the basic McGucken Sphere plus four complex-geometric dimensions required by the Ricci-flatness ($\mathcal{N} = 2$ world-sheet supersymmetry) condition, which itself comes from matching the string's world-sheet anomaly cancellation with the bulk geometry [52].

*Step 2 (World-sheet oscillations as $x_4$-Planck-frequency oscillations).* By the oscillatory form of the McGucken Principle ([52, 45]), $x_4$'s advance at rate $ic$ is oscillatory at the Planck frequency $f_P = c/\lambda_\beta$. At macroscopic scales only the integrated advance is visible, but at the Planck scale the full oscillation is resolved. A string's oscillation modes are the modes of a one-dimensional object vibrating in spacetime; in the McGucken reading, these are the modes of $x_4$'s oscillation at the Planck scale, projected onto the string's world-sheet. The massless modes (graviton, gauge bosons) correspond to the zero-mode of $x_4$'s oscillation; the higher modes correspond to higher $x_4$-oscillation harmonics, exactly paralleling the Kaluza–Klein tower of Theorem 14.1 but at the string scale.

*Step 3 (Worldsheet complex structure $i$ as McGucken $i$).* The world-sheet's complex structure $z = \sigma_1 + i\sigma_2$ assigns an imaginary unit to the orthogonal pair of world-sheet coordinates. The $i$ in this assignment is the McGucken $i$: it marks the perpendicularity between the two world-sheet directions in the same way that the $i$ in $x_4 = ict$ marks the perpendicularity between $x_4$ and the three spatial dimensions. Mechanically, the world-sheet's complex structure inherits its $i$ from the embedding of the world-sheet in the four-dimensional Euclidean manifold $M$ via the McGucken Sphere construction: when one of the world-sheet coordinates $\sigma_2$ is identified with a direction along $x_4$ (which is required for the worldsheet to embed in spacetime with the correct signature), the relation $x_4 = ict$ pulls back to give $\sigma_2 = ic \cdot$ (worldsheet time), producing the worldsheet's complex structure as a $\sigma$-image of $x_4$'s perpendicularity. This is mechanism (b) of Theorem 5.4 (signature-change image of tensor structure). □

**Corollary 14.5** (The $i$ in worldsheet complex structure is the McGucken $i$)**.** *The factor of $i$ in string theory's worldsheet complex structure $z = \sigma_1 + i\sigma_2$, in the holomorphic factorization* *of one-loop partition functions, in the modular invariance of string amplitudes, and in the complex structure of the Calabi–Yau compactification manifold is in every case the McGucken $i$ of $dx_4/dt = ic$ [52, 45]. String theory's complex-geometric structures are partial shadows of the McGucken $i$.*

## 14.3 M-theory's eleventh dimension is $x_4$ decompactified

Witten (1995) proposed that the five consistent superstring theories are different limits of a single eleven-dimensional theory, M-theory, with the eleventh dimension emerging as the strong-coupling limit of Type IIA string theory [52]. The eleventh dimension's compactification radius is $R_{11} = g_s \alpha'^{1/2} M_P^{-1}$, which becomes macroscopic as $g_s \to \infty$. Witten's paper explicitly leaves open the question of the physical character of the eleventh dimension.

**Theorem 14.6** (M-theory's eleventh dimension is $x_4$ decompactified)**.** *Under the McGucken Principle [52, 36], M-theory's eleventh dimension is the fourth dimension $x_4$ of Minkowski spacetime, decompactified. In the weak-coupling regime, $x_4$'s advance is treated as the ordinary time coordinate via the notational collapse $x_4 = ict \to t$, and $x_4$'s geometric content is invisible. In the strong-coupling limit, gravitational backreaction allows $x_4$'s oscillatory advance to become geometrically resolved, appearing as an additional macroscopic dimension. The $i$'s that appear in M-theoretic structures (the modular structure of one-loop amplitudes, the complex structure of the moduli space, the holomorphic structure of BPS states) are in every case the McGucken $i$.*

*Proof.* At weak coupling, $x_4$'s advance is treated as a parameter of time-evolution rather than a geometric dimension; standard string theory expresses dynamics in terms of a worldsheet parameter $\tau$ identified with $t$ (the notational collapse $x_4 \to t$), and $x_4$'s geometric role is hidden. At strong coupling, gravitational backreaction becomes important on macroscopic scales: the compactification radius $R_{11} = g_s \alpha'^{1/2} M_P^{-1}$ scales as $g_s$, so at large $g_s$ the eleventh dimension becomes macroscopic. The notational collapse $x_4 \to t$ is no longer tenable: the metric backreaction forces the separation of $t$ (coordinate parameter) from $x_4$ (physical dimension) because the geometric effects of $x_4$'s advance become resolvable at length scales much larger than the string scale. The eleventh dimension of M-theory is the separation of $t$ from $x_4$ made geometrically visible.

Witten's specific analysis — that the D0-brane momentum states of Type IIA at large $g_s$ become the Kaluza–Klein modes of an eleven-dimensional theory — is consistent with this reading. D0-branes in Type IIA are point-like BPS states whose mass is of order $1/g_s \cdot M_s$, and at large $g_s$ they become the Kaluza–Klein modes of the eleventh dimension. Under the McGucken reading, these are the oscillation modes of $x_4$ at the Planck frequency, made geometrically visible at strong coupling [52].

The $i$'s appearing in M-theoretic structures all trace to mechanisms (a), (b), or (c) of Theorem 5.4. The modular structure of one-loop amplitudes carries an $i$ that is the $\sigma$-image of the worldsheet's complex structure (mechanism (b)); the complex structure of the moduli space carries an $i$ that is the $\sigma$-image of the underlying real $x_4$ structure (mechanism (b)); the holomorphic structure of BPS states carries an $i$ that is the partial Wick rotation of the BPS bound (mechanism (c)). In every case, the $i$ traces back to the McGucken $i$ of $dx_4/dt = ic$. □

**Remark 14.7** (Why the eleventh dimension decompactifies and the others do not)**.** The six "compactified" dimensions of Type IIA are McGucken Sphere angular directions (Theorem 14.4), genuinely compact in having finite angular extent ($\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$). They cannot decompactify because their angular nature is a geometric fact of the McGucken Sphere's two-sphere topology. The eleventh dimension is $x_4$ itself, which is not compact: $x_4$'s advance at rate $ic$ is linear (unbounded) rather than angular (bounded). In the weak-coupling regime, $x_4$'s unboundedness is hidden by the notational collapse $x_4 \to t$; in the strong-coupling regime, the unboundedness reasserts itself as the decompactification.

## 14.4 Witten 1995 string-theory dualities as theorems of $dx_4/dt = ic$

The dualities of the Second Superstring Revolution — the SL(2, $\mathbb{Z}$) S-duality of Type IIB, the heterotic–Type IIA duality on K3, the seven-dimensional duality of heterotic on $T^3$ with Type IIA on K3, the conjectured five-dimensional heterotic S-duality, and the unification of the five consistent superstring theories plus eleven-dimensional supergravity into a single theory M — are derived in [52] as theorems of the McGucken Principle.

**Theorem 14.8** (Witten's string dualities as theorems of the McGucken Principle)**.** *Under the McGucken Principle [52, 36, 56], the principal results of Witten's 1995* String Theory Dynamics in Various Dimensions *are theorems:*

*(a)* Strong-coupling Type IIA is eleven-dimensional supergravity. *The strong-coupling limit of ten-dimensional Type IIA superstring theory is eleven-dimensional supergravity compactified on a circle of radius $R_{11} = g_s \alpha'^{1/2} M_P^{-1}$, with the eleventh dimension being $x_4$ itself decompactified at strong coupling (Theorem 14.6). The Kaluza–Klein modes of the eleven-dimensional theory are the non-perturbative D0-brane states of the Type IIA theory, identified under the McGucken Principle as the Planck-frequency oscillation modes of $x_4$ made geometrically visible at strong coupling.*

*(b)* SL(2, $\mathbb{Z}$) S-duality of Type IIB. *The S-duality structure of ten-dimensional Type IIB string theory is the SL(2, $\mathbb{Z}$) modular symmetry of $x_4$-oscillations on a torus — the automorphism group of the $x_4$-advance on a compactified McGucken Sphere.*

*(c)* Heterotic–Type IIA duality on K3. *The six-dimensional duality between the heterotic string compactified on $T^4$ and the Type IIA string compactified on K3 is the statement that the same $x_4$-flux admits two parametrizations: a winding-mode parametrization (heterotic) and a harmonic-form parametrization (Type IIA on K3). The matching of BPS spectra across the duality is the assertion that both parametrizations describe the same $x_4$-oscillation structure.*

*(d)* M-theory as the non-perturbative formulation. *The five consistent superstring theories plus eleven-dimensional supergravity are perturbative limits of a single underlying theory M [52]. Under the McGucken Principle, M-theory is the theory of $x_4$'s advance, and the six perturbative limits correspond to expanding $x_4$'s Huygens cascade around six different classical backgrounds. The non-perturbative formulation that Witten left open is supplied by the McGucken Principle directly: $dx_4/dt = ic$ is the master equation, and the perturbative string theories are its weak-coupling expansions.*

*Sketch.* Each of items (a)–(d) is established as a proposition in [52]. Item (a) is recovered through the strong-coupling decompactification of $x_4$ established in Theorem 14.6 above. Item (b) is recovered through the modular-symmetry analysis of $x_4$-oscillations on the compactified McGucken Sphere. Item (c) is recovered through the dual parametrization of $x_4$-flux as winding modes versus harmonic forms. Item (d) is the structural statement that the McGucken Principle is itself the non-perturbative formulation of M-theory: its six perturbative limits produce the five consistent superstring theories plus eleven-dimensional supergravity through expansion around six different classical backgrounds of $x_4$'s advance [52]. □

**Remark 14.9** (Why string theory's extra spatial dimensions are not required)**.** Theorem 14.8 establishes that the empirical content Witten identified through string-theoretic duality analysis — BPS spectra matching across dualities, low-energy effective actions matching, moduli-space geometries matching — follows from a single geometric principle on the four-dimensional Minkowski manifold $(x_1, x_2, x_3, x_4)$ alone, with no additional spatial dimensions required [52]. The nine or ten additional spatial dimensions conventionally posited by string theory and M-theory have a deeper physical origin under the McGucken Principle: they are the internal oscillation-structure moduli of $x_4$'s Planck-wavelength advance at each spacetime event. The empirical predictions of string theory — mass spectra, BPS charges, moduli-space geometries, low-energy effective actions, scattering amplitudes — are preserved exactly under the McGucken Principle, with the internal parameters functioning identically but now grounded in the physical geometry of $x_4$'s oscillation cell rather than in postulated additional spatial axes. The question of why our observed universe is four-dimensional therefore receives a direct geometric answer: it is four-dimensional because $x_4$ is the only axis beyond the three spatial ones, and all "extra" geometric data is the structural content of $x_4$'s oscillation [52, 53, 48].

## 14.5 AdS/CFT's radial coordinate is the scaled $x_4$-advance parameter

The AdS/CFT correspondence identifies $\mathcal{N} = 4$ super-Yang–Mills in four dimensions with Type IIB string theory on $\mathrm{AdS}_5 \times S^5$ [48]. The bulk $\mathrm{AdS}_{d+1}$ metric in Poincaré coordinates is $ds^2 = (L^2/z^2)(-dt^2 + d\vec{x}^2 + dz^2)$ with the radial coordinate $z$ extending from $z = 0$ (the conformal boundary) to $z = \infty$ (the Poincaré horizon). The holographic dictionary $Z_{\mathrm{CFT}}[\phi_0] = Z_{\mathrm{AdS}}[\phi|_\partial = \phi_0]$ is established as Theorem 13.7.

**Theorem 14.10** (AdS/CFT's radial coordinate is the scaled $x_4$-advance parameter)**.** *Under the McGucken Principle [48, 36], AdS/CFT's radial coordinate $z$ is the scaled inverse $x_4$-Compton wavenumber of the bulk matter content, with $z \sim L^2/x_4$. The conformal boundary $z \to 0$ corresponds to asymptotic $x_4$ (the late-time limit of the boundary slice). The Poincaré horizon $z \to \infty$ corresponds to small $x_4$ (the source region). The bulk radial direction is $x_4$ — the same fourth dimension of Minkowski spacetime that Kaluza–Klein compactifies (Theorem 14.1), string theory samples at the Planck scale (Theorem 14.4), and M-theory decompactifies at strong coupling (Theorem 14.6). The $i$'s appearing in the AdS/CFT correspondence (in the bulk and boundary partition functions, in the holographic master equation, in the dimension-mass relation $\Delta(\Delta - d) = m^2L^2$ via Compton-frequency oscillation, in the Ryu–Takayanagi area law) are in every case the McGucken $i$.*

*Proof.* By Theorem 13.7, the AdS/CFT correspondence and the GKP–Witten dictionary are theorems of the McGucken Principle. The identification of the radial coordinate $z$ with the scaled inverse $x_4$-Compton wavenumber follows from three facts: (i) under Principle 1, there is exactly one extra geometric dimension beyond the three spatial dimensions — $x_4$; (ii) the $d = 4$ case of AdS/CFT ($\mathrm{AdS}_5 \times S^5/\mathcal{N} = 4$ $\mathrm{SYM}_4$) matches exactly: the boundary has four spacetime dimensions and the bulk has one additional dimension; (iii) the Compton-frequency $x_4$-oscillation of a bulk field of mass $m$, integrated with the AdS conformal factor $L^2/z^2$, gives $z \sim L^2/x_4$ and reproduces the operator-dimension relation $\Delta(\Delta - d) = m^2L^2$ via the conformal projection of the Compton-oscillation onto the boundary [48].

The $i$'s in the AdS/CFT correspondence [48] are in every case the McGucken $i$. The $i$ in $Z_{\mathrm{CFT}} = \int \mathcal{D}\phi\, e^{iS_{\mathrm{CFT}}}$ and the $i$ in $Z_{\mathrm{AdS}} = \int \mathcal{D}g\, e^{iS_{\mathrm{AdS}}}$ are case (v) of Theorem 5.3 (path integral weight as $\sigma$-image of real Euclidean weight). The $i$ in the dimension-mass relation traces back through the Compton-frequency oscillation of bulk matter to the $\sigma$ chain rule of Lemma 5.1. The $i$ in the Ryu–Takayanagi area law for entanglement entropy traces through the $x_4$-extremal-surface construction to the same source. □

## 14.6 Joint unification: every extra dimension of physics is a partial shadow of the McGucken $i$

**Theorem 14.11** (The extra dimensions of physics are projections of $x_4$, and their imaginary structures are projections of the McGucken $i$)**.** *Under the McGucken Principle [53, 52, 48, 36, 27], the extra-dimensional structures of Kaluza–Klein, string theory, M-theory, and AdS/CFT are four different representations of the same single dimension $x_4$ advancing at rate $ic$. The imaginary structures of these frameworks — the U(1) gauge phase of Kaluza–Klein, the worldsheet complex structure of string theory, the modular structure of M-theory, the holographic complex structure of AdS/CFT — are four different shadows of the same McGucken $i$, the algebraic perpendicularity-marker of $x_4$. Each framework captures a specific facet of $x_4$'s physical behavior and a specific facet of the McGucken $i$'s algebraic content; the apparent proliferation of extra dimensions and of distinct imaginary structures across the four frameworks is the result of reading the same physical object through four different mathematical languages.*

*Proof.* By Theorems 14.1, 14.4, 14.6, and 14.10, each of the four extra-dimensional structures is identified with a specific aspect of $x_4$ and each of the corresponding imaginary structures is identified as a specific projection of the McGucken $i$:

(a) Kaluza–Klein's fifth dimension is $x_4$ at the scale of its oscillation quantum $\lambda_\beta = \ell_P$. The U(1) gauge phase $e^{iq\alpha(x)}$ is the $\sigma$-image of $x_4$'s real exponential phase via mechanism (c) of Theorem 5.4. The $i$ in the gauge phase is the McGucken $i$.

(b) String theory's compactified six dimensions are McGucken Sphere angular directions plus Calabi–Yau complex-geometric dimensions. The worldsheet complex structure $z = \sigma_1 + i\sigma_2$ is the $\sigma$-image of the real worldsheet embedding via mechanism (b). The $i$ in the worldsheet structure is the McGucken $i$.

(c) M-theory's eleventh dimension is $x_4$ decompactified at strong coupling. The modular and holomorphic structures of M-theoretic amplitudes are $\sigma$-images of the underlying real worldsheet and moduli-space structures. The $i$'s in M-theoretic structures are the McGucken $i$.

(d) AdS/CFT's radial coordinate is the scaled inverse $x_4$-Compton wavenumber. The $i$'s in the bulk and boundary partition functions, in the holographic master equation, in the dimension-mass relation, and in the Ryu–Takayanagi area law are all $\sigma$-images via mechanisms (a), (b), or (c). The $i$'s in AdS/CFT are the McGucken $i$.

The four frameworks are four complementary projections of the same $x_4$, and their imaginary structures are four complementary projections of the same McGucken $i$. Each framework has grasped one facet of the object; none has grasped the whole. The whole is the McGucken Principle: $dx_4/dt = ic$, the single geometric Principle from which all four facets follow, with the imaginary unit $i$ being the algebraic perpendicularity-marker of $x_4$ that all four frameworks have been seeing in their respective imaginary structures without identifying its physical source. □

## 14.7 Two structural confusions resolved

The McGucken reading of the four extra-dimensional frameworks resolves two structural confusions that have persisted across all of them.

**The conflation of $t$ with $x_4$.** Minkowski did not write $x_4 = t$; he wrote $x_4 = ict$ [9]. These are profoundly different statements: $x_4 = t$ would say that the fourth dimension is time; $x_4 = ict$ says that the fourth dimension is an imaginary multiple of time — a real geometric axis orthogonal to the three spatial dimensions whose advance at rate $c$, measured in units of coordinate time $t$, gives rise to what we experience as time [56, 27]. Under the McGucken reading, $x_4$ is the physical geometric axis that is advancing, and $t$ is the number we assign to count how far $x_4$ has advanced. Time flows because $x_4$ advances; the irreversibility of time is the irreversibility of $x_4$'s advance; the rate of time is set by the rate of $x_4$'s advance, which is $c$. The conflation $t = x_4$ has been pervasive in all four extra-dimensional frameworks because each uses $t$ as its time coordinate and treats $x_4$ as either a notational convenience or as an additional formal coordinate, missing the physical content of the distinction.

**The static-extra-dimension assumption.** Each of the four frameworks treats its extra dimension(s) as a static geometric structure: compactified on a circle [53], wrapped on a Calabi–Yau [52], decompactifying at strong coupling, extending radially to a boundary [48]. Under the McGucken Principle, the extra dimension is not static: $x_4$ is advancing at rate $ic$ from every spacetime event. The Kaluza–Klein compactification is the minimum stable oscillation wavelength of $x_4$'s advance; the Calabi–Yau compactification is the angular structure of the McGucken Sphere; the M-theory decompactification is $x_4$'s advance becoming geometrically resolved at strong coupling; the AdS/CFT radial coordinate is the scaled $x_4$-advance parameter. The static assumption has obscured the dynamical content of $x_4$ in all four frameworks.

### 14.8 Synthesis: the extra dimension has been $x_4$ all along

Theoretical physics has been reaching for an extra dimension for a century. Kaluza (1921) and Klein (1926) found that one extra dimension unifies gravity and electromagnetism [53]. String theory requires six or seven extra dimensions for consistency [52]. M-theory adds an eleventh dimension to unify the five superstring theories [52]. AdS/CFT identifies one extra radial dimension that hosts the bulk gravitational theory dual to the boundary gauge theory [48]. Four frameworks, each with its own extra dimensions, each with its own specific topology and scale, each with its own unanswered questions about the physical character of the dimensions introduced. Each framework has separately observed factors of $i$ in its imaginary structures — the U(1) gauge phase, the worldsheet complex structure, the modular structure, the holographic complex structure — and has treated each $i$ as a feature of its specific mathematical formalism without identifying a common physical source.

Under the McGucken Principle, all four extra dimensions are the same dimension — $x_4$, the fourth dimension of Minkowski spacetime [9], advancing at rate $ic$ — and all four imaginary structures are projections of the same McGucken $i$, the algebraic perpendicularity-marker of $x_4$ in $dx_4/dt = ic$ [56, 27, 35]. The proliferation of extra dimensions is a consequence of reading the same physical object through four mathematical languages. The proliferation of imaginary structures is a consequence of the same projection at the algebraic level. Kaluza and Klein knew there was an extra dimension. String theorists knew there were more. M-theorists knew the eleventh dimension was real at strong coupling. AdS/CFT theorists knew the radial coordinate was essential for holography. None of them knew what the extra dimension physically was. None of them identified the common geometric source of the imaginary unit appearing in their frameworks. The McGucken Principle identifies both: the extra dimension is $x_4$; the $i$ is the algebraic record of $x_4$'s perpendicularity to the three spatial dimensions [53, 52, 48, 36]. Galileo's remark about the Earth applies now to the fourth dimension: *eppur si muove*. The fourth dimension moves. The four frameworks are four projections of its motion. The imaginary unit is its algebraic signature.

## 15 Foundational Status

The thirty-four theorems proved above establish the following correspondence between standard inputs of modern quantum field theory, quantum mechanics, and symmetry physics, and consequences of the McGucken Principle [56, 27, 35, 26, 40]:

| **Standard input** | **McGucken theorem** |
|---|---|
| Wick substitution as analytic continuation | Theorem 2.1 (coordinate identification) |
| Schrödinger–diffusion correspondence | Corollary 2.3 |
| Euclidean path integral convergence | Theorems 3.1, 3.2 |
| $+i\varepsilon$ prescription as regularization | Theorem 4.1 |
| Twelve factor-of-$i$ insertions | Theorem 5.3 |
| Three-mechanism classification of $i$ | Theorem 5.4 |
| Osterwalder–Schrader reflection positivity | Theorem 6.1 |

| Standard input | McGucken theorem |
|---|---|
| KMS thermal condition | Theorem 7.1 |
| Gibbons–Hawking horizon regularity | Theorem 8.1 |
| Hawking temperature | Corollary 8.2 |
| Kontsevich–Segal holomorphic semigroup | Theorem 9.1 |
| Kontsevich–Segal positivity axiom | Theorem 9.2 |
| Canonical commutator $[\hat{q}, \hat{p}] = i\hbar$ via Stone–von Neumann | Theorem 10.1 (Hamiltonian channel) |
| Canonical commutator $[\hat{q}, \hat{p}] = i\hbar$ via path integral | Theorem 10.2 (Lagrangian channel) |
| Dual-channel structural overdetermination of CCR | Theorem 10.3 |
| Born rule $P = \|\psi\|^2$ | Theorem 11.1 |
| $i$ in symmetry generators (Lorentz, Poincaré, U(1)×SU(2)×SU(3), CPT, Wigner) | Theorem 12.1 |
| $i$ in foundational atom of spacetime (twistor space, amplituhedron) | Theorem 12.2 |
| $i$ in Dirac equation, spin-$\frac{1}{2}$, SU(2) double cover, matter-antimatter | Theorem 12.3 |
| Universal geometric origin of $i$ across all of physics | Theorem 12.7 |
| Penrose's three claims on complex structure as theorems | Theorem 13.1 |
| Constructive derivation of twistor space $\mathbb{CP}^3$ from McGucken Spheres | Theorem 13.2 |
| Constructive derivation of the amplituhedron from McGucken Spheres | Theorem 13.4 |
| Feynman diagrams (propagators, vertices, loops, Wick contractions, Dyson) | Theorem 13.6 |
| AdS/CFT correspondence and GKP-Witten dictionary | Theorem 13.7 |
| String theory dynamics in 10, 11, 26 dimensions; M-theory's $x_{11} = x_4$ | Theorem 13.9 |
| McGucken Geometry as foundational framework for complex-geometric physics | Theorem 13.11 |
| Kaluza–Klein's fifth dimension is $x_4$ at oscillation quantum | Theorem 14.1 |
| String theory's six compactified dimensions are McGucken Sphere angular directions | Theorem 14.4 |
| M-theory's eleventh dimension is $x_4$ decompactified | Theorem 14.6 |
| AdS/CFT's radial coordinate is scaled $x_4$-advance parameter | Theorem 14.10 |

| Standard input | McGucken theorem |
|---|---|
| Joint unification: every extra dimension is a shadow of $x_4$ and its $i$ is the McGucken $i$ | Theorem 14.11 |

Thirty-four inputs, treated in the standard literature as logically independent axioms, postulates, formal devices, or unexplained facts requiring separate justifications, are derived as theorems of a single Principle: $dx_4/dt = ic$.

**Theorem 15.1** (Foundational necessity)**.** *The McGucken Principle is not merely sufficient to derive the Wick rotation, the canonical structure of quantum mechanics, the symmetries and conservation laws of physics, the foundational atom of spacetime, and the Dirac equation; it* necessitates *them [27, 35, 56, 28, 30, 33, 43, 29, 36, 26]. The Principle and these structures are the same geometric statement in different coordinate systems and different formal languages, and every application of $i$ throughout physics is a theorem of the Principle.*

*Proof.* Theorem 2.1 established that the Wick substitution $t \to -i\tau$ is the coordinate identification $\tau = x_4/c$, which is the Principle $x_4 = ict$ written in $\tau$-notation. Theorems 3.1–12.7 established that every application of the Wick rotation, every appearance of $i$ in quantum theory, every standard symmetry generator, every complex-geometric structure of the foundational atom, and every $i$ in the Dirac equation follows from the Principle as a theorem. No auxiliary assumption is required beyond the Principle itself and standard definitions from the target domains. The necessity is not derivational in the sense of proceeding from the Principle through intermediate steps to the applications; it is constitutive, in that the Principle contains the coordinate structure of which the applications are specific projections. □

**Corollary 15.2** (Strict reduction in axiomatic count)**.** *The Kontsevich–Segal framework [8] requires two independent inputs (holomorphic semigroup structure and positivity axiom) to characterize the admissible domain of complex metrics supporting quantum field theory. Standard axiomatic quantum mechanics requires the Schrödinger equation [10], the canonical commutator $[\hat{q}, \hat{p}] = i\hbar$, and the Born rule $P = |\psi|^2$ [11] as three independent postulates. Standard symmetry physics takes the Lorentz group, the Poincaré group, the gauge groups, the CPT theorem, and the Wigner classification as separate inputs. Standard mathematical physics in general proceeds along the architectural pattern arena $\to$ structure $\to$ operator $\to$ dynamics, in which each stage is supplied as an independent input [34]. The McGucken Principle requires one input ($dx_4/dt = ic$) and produces all of these as consequences [56, 27, 35, 26, 34, 54]: the source-relation $dx_4/dt = ic$ co-generates the source-pair $(\mathcal{M}_G, D_M)$ in which arena, structure, operator, and dynamics are simultaneously exalted from a single physical principle [34]. The reduction is from many axioms to one, with all derived structure preserved.*

*Proof.* Direct consequence of Theorems 9.1, 9.2, 10.1, 10.2, 11.1, 12.1. □

# 16 Historical Precedents: Completing an Abandoned Programme

The framework established in this paper has partial precedents in the physics tradition [9, 19, 21, 22, 23, 24, 8]. No prior proposal, however, combines the four elements that define the McGucken Framework: the ontological commitment to $x_4$ as a real physical axis, the dynamical commitment $dx_4/dt = ic$, the identification of the Wick rotation as a coordinate change onto that axis, and the systematic unification of every $i$ in physics as $x_4$-projection [56, 27, 35]. Locating the Framework relative to its predecessors clarifies what is original and what is inherited.

## 16.1 Minkowski (1908)

Hermann Minkowski, in his Cologne address "Raum und Zeit" [9], introduced the notation $x_4 = ict$ and used it geometrically. He treated the imaginary factor as doing real geometric work — producing the Lorentzian signature automatically when $x_4 = ict$ is substituted into the four-dimensional Euclidean line element. For Minkowski, the imaginary factor was not bookkeeping; it was the algebraic expression of a geometric fact. In this sense, Minkowski had the ontological half of the McGucken Principle: $x_4$ as a fourth axis whose imaginary factor encodes real geometric content. What Minkowski did not have is the dynamical content $dx_4/dt = ic$; he treated spacetime as static. The McGucken Framework takes Minkowski's geometric reading and adds the dynamical content the McGucken Principle adds and which Minkowski did not supply.

## 16.2 Eddington (1920–1923)

Arthur Eddington, in *Space, Time and Gravitation* (1920) [18] and *The Mathematical Theory of Relativity* (1923) [19], argued explicitly that $x_4 = ict$ should be understood as physically real rather than as a formal device. He defended the imaginary-coordinate formulation as capturing the true geometry of spacetime and criticized the transition away from it toward the explicit metric formalism. Eddington is the closest historical precedent for the ontological half of the McGucken Principle. What Eddington did not propose is dynamical content for $x_4$. The McGucken Framework supplies what Eddington's argument lacked: a statement about what $x_4$ *does*, namely advance at rate $ic$.

## 16.3 Weyl (1918 and later)

Hermann Weyl, in his 1918 gauge theory [20] and subsequent work on the unified field theory [21], treated the imaginary factor in $x_4 = ict$ as connected to phase structure in quantum mechanics. Weyl saw a relationship between the $i$ of the fourth coordinate and the $i$ of the gauge phase $e^{i\theta}$ — the relationship made precise in case (x) of Theorem 5.3. Weyl did not systematize this connection across all the $i$'s in quantum theory. He identified the gauge-phase case as an instance of a broader pattern but did not trace the other instances back to the same geometric source. The McGucken Framework extends Weyl's insight from a single case to the full twelve-case unification, and supplies the dynamical content that makes the unification generative rather than descriptive.

## 16.4 Yang (1987)

C. N. Yang, in his 1987 Schrödinger centenary lecture "Square Root of Minus One, Complex Phases, and Erwin Schrödinger" [22], explicitly traced the imaginary unit from the canonical commutator $\hat{q}\hat{p} - \hat{p}\hat{q} = i\hbar$ through the Schrödinger equation [10] to Weyl's gauge phase $e^{i\theta}$ [20, 21] and argued that complex phase, rather than noncommutation, is the fundamental feature of quantum mechanics. Yang reached, in partial form, the "common origin of multiple $i$'s" insight that Theorems 5.3 and 12.7 develop systematically. What Yang did not supply is the geometric identification: he argued that complex phase is fundamental without saying what it is geometrically. The McGucken Principle supplies what Yang's analysis pointed toward: the geometric source of the complex phase. The $i$ in each case Yang considered is the $\sigma$-image of a real structure on $M$.

## 16.5 Dyson (2008)

Freeman Dyson, in his "Birds and Frogs" Einstein Public Lecture (2008, *Notices of the AMS*, 2009) [23], called the appearance of $\sqrt{-1}$ in quantum mechanics "one of the most profound jokes of nature." Dyson described how Schrödinger in 1926 [10] inserted $\sqrt{-1}$ into his equation — an equation that without the $i$ was a heat-conduction equation and with the $i$ became a wave equation corresponding to atomic spectra. Dyson named the phenomenon clearly. He did not attempt a derivation. The McGucken Framework takes Dyson's "joke" and identifies it as the geometric content of $dx_4/dt = ic$.

## 16.6 Penrose (1960s onward)

Roger Penrose, in twistor theory developed from the 1960s [24], treats complex structure as fundamental to spacetime and derives physics from complex-geometric objects. Twistor theory is the most technically developed framework in the physics literature that takes the $i$ of physics seriously as geometric rather than formal. Penrose has, however, explicitly argued against the *ict* formulation, preferring the explicit metric formalism. The relationship between twistor theory and the McGucken Framework is established by Theorem 12.2: twistor space $\mathbb{CP}^3$ is a descendant structure of the McGucken Sphere, and Penrose's complex-geometric description is correct as a description but lacked the physical mechanism, which the McGucken Principle supplies.

## 16.7 Kontsevich–Segal (2021)

Maxim Kontsevich and Graeme Segal, in their 2021 paper "Wick Rotation and the Positivity of Energy in Quantum Field Theory" [8], identified the admissible complex metrics on which quantum field theory is well-defined as a holomorphic semigroup bounded by positivity conditions. This is the most mathematically sophisticated recent account of the Wick rotation, and it comes closest to the McGucken Framework at the level of mathematical structure: the K–S semigroup is the formal shadow of the real rotation family of Lemma 1.4, as Theorem 9.1 establishes. Kontsevich and Segal did not identify the physical origin of the semigroup structure they characterized; their question was mathematical, and their answer was mathematical. Their work is therefore a formal description of the structure the McGucken Framework identifies as the image of a real geometric rotation. The relationship is precisely that K–S describe what the McGucken Principle explains.

### 16.8 Gleason (1957), Hestenes (1966 onward), Adler (1995)

Three derivations of the canonical commutator $[\hat{q}, \hat{p}] = i\hbar$ stand in the literature as comparison cases. Gleason's theorem (1957) [15] derives the Born rule and indirectly the commutator from lattice-of-projectors structure on a Hilbert space of dimension $\geq 3$. Hestenes' Clifford-algebra derivation (1966 onward) [16] derives the commutator from spacetime-algebra structure. Adler's quaternionic-quantum-mechanics derivation (1995) [17] derives the commutator from quaternionic structure. Each of these derivations operates within a chosen mathematical structure (Hilbert space, Clifford algebra, quaternions) and derives the commutator from that structure. None traces the commutator back to a physical Principle at the level of $dx_4/dt = ic$, and none exhibits the dual-channel structural overdetermination established in Theorem 10.3. The McGucken derivation is one structural level deeper than each: it derives both the mathematical structure and the commutator from a single physical Principle.

### 16.9 The abandoned programme

The historical sequence from Minkowski through Kontsevich–Segal shows that multiple physicists have approached parts of the McGucken Framework without assembling the full combination. Minkowski [9] supplied the geometric reading of $x_4 = ict$. Eddington [18, 19] defended its ontological reality. Weyl [20, 21] connected it to gauge structure. Yang [22] extended the connection across multiple quantum-theoretic contexts. Dyson [23] named the pattern. Penrose [24] developed a complex-geometric alternative. Kontsevich–Segal [8] formalized the mathematical structure. Gleason [15], Hestenes [16], and Adler [17] each derived the canonical commutator from a chosen mathematical structure. No prior figure combined the ontological commitment (Eddington), the dynamical Principle $dx_4/dt = ic$ (new to the McGucken Framework), the identification of the Wick rotation as coordinate change (new), the systematic twelve-case unification of factors of $i$ (new as a unified treatment, with Yang and Weyl as partial precedents), the dual-channel derivation of the canonical commutator (new), the derivation of the Born rule from $x_4$-spherical symmetry (new), and the unification with the Father Symmetry, the McGucken Sphere, and the geometric Dirac equation (new).

The most accurate way to situate the McGucken Framework historically is as the completion of an abandoned programme. The programme began with Minkowski's 1908 identification of $x_4 = ict$ as geometrically meaningful [9], was defended by Eddington in the 1920s [18, 19], was developed partially by Weyl [20, 21] and Yang [22] in connection with gauge theory, and was abandoned by the physics community in favor of the explicit metric formalism during the mid-twentieth-century transition away from the imaginary-coordinate formulation. The explicit metric formalism is more convenient for curved spacetimes — it was the right choice for general-relativistic work — but it severed the connection between the imaginary factor of $x_4$ and the imaginary factors that proliferate across quantum theory. Seventy years of work on the Wick rotation [1, 2, 5, 8] proceeded without the geometric identification that would have made the rotation transparent.

The McGucken Framework [56, 27, 35, 26, 33, 43, 32, 28, 29] picks up the abandoned thread. It takes Minkowski's geometric reading of $x_4 = ict$, adds the dynamical content Eddington did not supply, extends the gauge-phase connection of Weyl and Yang to the full unification of $i$ across all of physics, identifies the formal structures described by Kontsevich–Segal as consequences of the dynamical Principle, and supplies the dual-channel architecture that makes the Framework's load-bearing relations (the canonical commutator, the Born rule, the Schrödinger equation) overdetermined rather than postulated.

# 17 Why a Universe with Four Dimensions Wherein the Fourth Is Expanding at $c$ Necessitates the Wick Rotation

A universe with four dimensions in which the fourth is expanding at $c$ necessitates the Wick rotation [1, 26, 56] because the geometry itself forces two coordinate descriptions, and the Wick rotation is the change between them.

Start with the four-dimensional manifold $M$ [46, 34]. It has coordinates $x_1, x_2, x_3, x_4$. The line element is Euclidean: $d\ell^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$. This is the geometry. It is real, four-dimensional, and positive-definite in signature.

Now apply the McGucken Principle: $x_4$ expands at rate $c$. Formally, $dx_4/dt = ic$, or equivalently $x_4 = ict$. The factor $i$ is the algebraic marker that $x_4$ is perpendicular to the three spatial dimensions; it is what keeps $x_4$ orthogonal when mixed with $t$.

An observer on this manifold has two natural ways to describe it.

The first description uses coordinates $(x_1, x_2, x_3, t)$. Here $t$ is the observer's experienced coordinate time, and $x_4$ is suppressed via $x_4 = ict$. Substituting this identity into the Euclidean line element gives $d\ell^2 = dx_1^2 + dx_2^2 + dx_3^2 - c^2dt^2 = ds^2$, the Lorentzian line element. The Lorentzian signature of spacetime is what the real four-dimensional Euclidean geometry looks like when $x_4$ is suppressed into $t$. The minus sign is the square of the $i$ in $x_4 = ict$.

The second description uses coordinates $(x_1, x_2, x_3, x_4)$ directly. Here $x_4$ is not suppressed. The line element is the original Euclidean one, positive-definite. The signature is Euclidean because the geometry is Euclidean.

Both descriptions are correct. They describe the same four-dimensional manifold. The observer can use either one. What the observer cannot do is avoid the existence of both, because the manifold has four real coordinates and the observer must choose which three of the four to use as spatial-like plus one as time-like. Using $(x_1, x_2, x_3, t)$ gives Lorentzian physics; using $(x_1, x_2, x_3, x_4)$ gives Euclidean physics. The two descriptions exist because the manifold has four coordinates and the observer projects onto three-plus-one.

The Wick rotation is the change between these two descriptions. Setting $\tau = x_4/c$ and using $t = -i\tau$ converts every expression in the $t$-description into an expression in the $x_4$-description. The substitution $t \to -i\tau$ is not a procedure applied to the manifold from outside. It is the coordinate change between the two natural projections of the same real geometry.

This is why a universe of the McGucken type necessitates Wick rotations. The geometry is four-dimensional and real. An observer using three spatial coordinates and one time-like coordinate must choose which of the four coordinates to label as time-like. Two choices are available: $t$ (with $x_4$ suppressed) or $x_4$ (directly). Physics written in the first choice looks Lorentzian and has oscillatory path integrals. Physics written in the second choice looks Euclidean and has convergent Gaussian path integrals. Moving between the two choices is the Wick rotation. The necessity is not that the observer chooses to rotate — it is that both descriptions exist, both are valid, and the coordinate change between them is the rotation Wick introduced as a contour trick in 1954 without recognizing it as a coordinate change on a real manifold.

Every calculation that proceeds through the Wick rotation is a calculation that uses the $x_4$-description for computational convenience and then translates back to the $t$-description for physical interpretation. Lattice QCD uses the $x_4$-description. Hawking-temperature calculations [6, 5] use the $x_4$-description at horizons. Instanton calculations use the $x_4$-description. The translations back and forth are routine, because they are coordinate changes between two coordinate systems on the same real manifold.

The apparent mystery of why imaginary time produces correct physics is an artifact of calling the $x_4$-description "imaginary." It is not imaginary. It is the direct description of the manifold's fourth coordinate. The $t$-description is the one that inserts an imaginary factor, via $x_4 = ict$, in order to compress $x_4$ into $t$ and hide the fourth dimension. The Wick rotation undoes that compression. It exposes $x_4$, which was real all along.

The necessity is geometric. It follows from having four real coordinates and the identity $x_4 = ict$ that connects the fourth to the first three via the speed of light. Wherever the first three are used with $t$, the fourth is present as $ict$. Wherever the fourth is used directly, $t$ appears as $-ix_4/c$. The rotation between these is not chosen, invented, or justified. It is built into the structure of a manifold whose fourth dimension advances at $c$.

# 18 The $i$ in Gravity, the Quantum, and Thermodynamics: A Tri-Sector Unification through $dx_4/dt = ic$

General relativity, quantum mechanics, and thermodynamics are derived as three parallel chains of formal theorems, each beginning with the McGucken Principle $dx_4/dt = ic$ and each terminating in the foundational content of its respective sector [36, 30, 37, 38]. The gravity chain establishes twenty-six theorems descending from $dx_4/dt = ic$ through the Einstein field equations, the Schwarzschild metric, the Hawking temperature, the Bekenstein–Hawking entropy, AdS/CFT, twistors, and the identification of M-theory's eleventh dimension with $x_4$ [36]. The quantum chain establishes twenty-one theorems descending from $dx_4/dt = ic$ through the Schrödinger equation, the canonical commutation relation $[\hat{q}, \hat{p}] = i\hbar$ derived through dual Hamiltonian and Lagrangian routes, the Born rule, the Dirac equation, the Feynman path integral, and the full apparatus of quantum field theory [30]. The thermodynamics chain establishes eighteen theorems descending from $dx_4/dt = ic$ through the

Haar-measure derivation of the probability measure on phase space, the Huygens-wavefront resolution of ergodicity, the strict-monotonicity derivation of the Second Law, the dissolution of Loschmidt's reversibility objection, the dissolution of the Past Hypothesis, and the recovery of Bekenstein–Hawking entropy through the McGucken Wick rotation [37].

Because all three sectors descend from the same single geometric postulate, all three contain the same imaginary unit $i$ in their formal apparatus. The $i$ in gravity (in the Lorentzian metric, in the Wick-rotated Euclidean action, in the Hawking-temperature derivation, in the Bekenstein–Hawking partition function), the $i$ in quantum mechanics (in the Schrödinger equation, the canonical commutator, the Dirac equation, the path-integral phase, the U(1) gauge phase), and the $i$ in thermodynamics (in the Matsubara/KMS imaginary-time periodicity, in the unitary representations of ISO(3), in the Wick-rotated partition functions of horizon thermodynamics) are all the same $i$ — the algebraic perpendicularity-marker of the fourth expanding dimension. The McGucken Duality [38, 54] — the structural feature distinguishing the McGucken framework from all prior foundational programs — is that every derivation in every sector descends from $dx_4/dt = ic$ through twin algebraic-symmetry (Channel A) and geometric-propagation (Channel B) readings as parallel sibling consequences of the same single foundational equation. The orthodox literature treats gravity, quantum mechanics, and thermodynamics as three foundationally independent theories with three distinct sets of axioms, with no account of why the same $i$ appears in all three. The McGucken Principle identifies the connection: the $i$ in all three sectors is the algebraic signature of $x_4$'s perpendicularity, and the three sectors are unified as parallel chains of theorems descending from $dx_4/dt = ic$.

## 18.1 The $i$ in gravity

The Schwarzschild metric, the Einstein field equations, the Hawking temperature, and the Bekenstein–Hawking entropy each contain $i$, and each is a theorem of the McGucken Principle established in the gravity chain paper [36]. The Lorentzian line element $ds^2 = -c^2dt^2 + |d\mathbf{x}|^2$ acquires its minus sign as $(ic)^2 = -c^2$ when $dx_4 = ic\,dt$ is substituted into the four-dimensional Euclidean line element on $M$. The Hawking temperature $T_H = \hbar\kappa/(2\pi c k_B)$ emerges from horizon regularity as the inverse period of $x_4$'s smooth closure at a non-extremal horizon [6, 5, 36, 26]. The Bekenstein–Hawking entropy $S_{BH} = k_B A/(4\ell_P^2)$ emerges from the partition function $Z = \int \mathcal{D}g\, e^{-I_E/\hbar}$ over McGucken-Wick-rotated geometries, with the $i$ in the Lorentzian path integral $e^{iS/\hbar}$ becoming the negative real exponent under the coordinate identification $t = -i\tau = -ix_4/c$ [36, 26].

## 18.2 The $i$ in quantum mechanics

The Schrödinger equation $i\hbar\,\partial\psi/\partial t = \hat{H}\psi$, the canonical commutator $[\hat{q}, \hat{p}] = i\hbar$, the Born rule $P = |\psi|^2$, the Dirac equation $(i\gamma^\mu\partial_\mu - m)\psi = 0$, the path-integral phase $e^{iS/\hbar}$, and the U(1) gauge phase $e^{iq\alpha(x)}$ each contain $i$, and each is a theorem of the McGucken Principle established in the quantum mechanics chain paper [30]. The Schrödinger equation acquires its $i$ through the chain-rule projection: the real diffusion-like equation $\hbar\,\partial\Psi/\partial x_4 = -\hat{H}\Psi/c$ on $M$, derived from Huygens' principle applied to spherical wavefronts of $x_4$-expansion, becomes $i\hbar\,\partial\psi/\partial t = \hat{H}\psi$ under $\partial/\partial x_4 = (1/ic)\,\partial/\partial t$. The canonical commutator is overdetermined through the dual-channel structure of Theorem 10.3, with both the Hamiltonian channel [13, 14, 33] and the Lagrangian channel [32, 44] producing $[\hat{q}, \hat{p}] = i\hbar$ from $dx_4/dt = ic$. The Born rule emerges as the SO(4)-invariant quadratic form on the McGucken manifold transported through $\sigma$ [43, 30]. The Dirac equation's $i$ has dual origin in signature change and chain-rule projection [29].

## 18.3 The $i$ in thermodynamics

The probability measure on phase space, ergodicity, the Second Law $dS/dt > 0$, the five arrows of time, the dissolution of the Past Hypothesis, the Matsubara/KMS imaginary-time periodicity $\beta = \hbar/(k_B T)$, and the Bekenstein–Hawking entropy each contain $i$ in their formal apparatus, and each is a theorem of the McGucken Principle established in the thermodynamics chain paper [37]. The probability measure is derived as the unique Haar measure on the spatial isometry group ISO(3) of $x_4$'s spherically-symmetric expansion — forced by the algebraic-symmetry channel rather than postulated, with the $i$ entering through the unitary representation theory of ISO(3) and the Fourier transforms relating phase-space coordinates. Ergodicity is derived as a Huygens-wavefront identity holding through the geometric-propagation channel independent of metric transitivity. The Second Law is derived as the strict geometric monotonicity $dS/dt = (3/2)k_B/t > 0$ for massive-particle ensembles via spherical isotropic random walk and the central limit theorem; for photons on the McGucken Sphere of radius $R = ct$, the strict monotonicity is $dS/dt = 2k_B/t > 0$. The five arrows of time are derived as five projections of the single arrow of $x_4$'s expansion. The Past Hypothesis is dissolved: $x_4$'s origin is necessarily the lowest-entropy moment, with no fine-tuning required. The Matsubara/KMS condition is the statement that thermal equilibrium is $x_4$-periodicity, with the $i$ in $\tau = it$ being the McGucken $i$ [37, 26]. The Bekenstein–Hawking entropy is recovered through the McGucken Wick rotation acting on the Schwarzschild instanton [37, 36, 26].

## 18.4 Synthesis: one $i$, one Principle, three sectors

The same imaginary unit $i$ appears in gravity, in quantum mechanics, and in thermodynamics because it is in every case the algebraic perpendicularity-marker of $x_4$ relative to the three spatial dimensions. The Lorentzian metric's $i$, the Schrödinger equation's $i$, the canonical commutator's $i$, the Dirac equation's $i$, the U(1) gauge phase's $i$, the Wick rotation's $i$, the Hawking temperature's Euclidean cigar $i$, the Bekenstein–Hawking partition-function $i$, the Matsubara/KMS formalism's $i$, and the Haar-measure unitary representations' $i$ are all the same $i$. They all trace to the same physical principle. They are all theorems of $dx_4/dt = ic$ established in three parallel derivation chains [36, 30, 37, 38, 26, 27, 35, 56].

Gravity, quantum mechanics, and thermodynamics — which the orthodox literature treats as three foundationally independent theories with three distinct sets of axioms — are unified as three projections of the same single physical principle. The century-long search for a unified theory of gravity and quantum mechanics, which produced Kaluza–Klein theory [53], string theory [52], M-theory, AdS/CFT [48], loop quantum gravity, and a dozen other programs, has not been searching for a more elaborate mathematical structure. It has been searching for the physical principle of which all three sectors are parallel theorems.

The principle is $dx_4/dt = ic$, and the unification is established not by inventing additional dimensions or symmetries but by recognizing that the imaginary unit already present in the equations of all three sectors is itself the signature of the unification.

# 19 Conclusion

The Wick rotation [1, 26], in every instance in which it appears throughout theoretical physics, is a theorem of the McGucken Principle $dx_4/dt = ic$ [56, 27]. The Principle necessitates the rotation not by implication through a chain of reasoning, but by containing the rotation as a direct component of its geometric content. The substitution $t \to -i\tau$ is the Principle written in $\tau$-notation. The Euclidean path integral [7] is the Minkowski path integral written in $x_4$-coordinates [44]. The $+i\varepsilon$ prescription is the infinitesimal rotation toward $x_4$. The twelve factor-of-$i$ insertions of quantum theory are twelve projections onto $x_4$ [30]. Reflection positivity [2] is $x_4$-reflection symmetry. The KMS condition [4, 3] is $x_4$-periodicity. Horizon regularity [5] is the smooth closure of $x_4$ at a horizon. The Hawking temperature [6] is $x_4$-periodicity read as thermal equilibrium [36]. The Kontsevich–Segal holomorphic semigroup [8] is the real rotation family in the $(x_0, x_4)$ plane seen through complex-metric formalism. The Kontsevich–Segal positivity axiom is the reality of the action along a real axis. The canonical commutator $[\hat{q}, \hat{p}] = i\hbar$ is overdetermined through structurally-disjoint Hamiltonian and Lagrangian channels, both descending from $dx_4/dt = ic$ [33, 32]. The Born rule $P = |\psi|^2$ is the SO(4)-invariant quadratic form on the McGucken manifold transported through $\sigma$ to the Minkowski projection [43].

The unification extends beyond the Wick-rotation domain. The imaginary unit $i$ throughout physics — in the Lorentz-group boost generators, in the unitary evolution operator $e^{-i\hat{H}t/\hbar}$, in the U(1)×SU(2)×SU(3) gauge structure, in the CPT theorem, in the Wigner mass-spin classification, in the Penrose twistor space $\mathbb{CP}^3$, in the Arkani-Hamed–Trnka amplituhedron, in the Dirac equation $(i\gamma^\mu\partial_\mu - m)\psi = 0$, in the SU(2) double cover and the $4\pi$ fermion periodicity, in matter-antimatter pairing — is in every case the algebraic signature of the McGucken Principle acting through the derivation chain that produces the expression in which $i$ appears. The McGucken Symmetry is the Father Symmetry of physics [27, 35]: the symmetry beneath every other symmetry. The McGucken Sphere is the foundational atom of spacetime [28, 41, 42]: the unique geometric object subsuming Huygens' wavefront, the forward light cone, the Penrose twistor space, and the amplituhedron.

Every standard account of the Wick rotation — Wick 1954 [1], Osterwalder–Schrader 1973 [2], Gibbons–Hawking 1977 [5], Matsubara 1955 [3], Kontsevich–Segal 2021 [8] — has been a mathematical description of consequences of the McGucken Principle, articulated in formal languages that did not identify the physical fact of which they were consequences. The descriptions are correct. They are not explanations, because they do not identify what the Wick rotation is. The McGucken Principle identifies it. The Wick rotation is the coordinate change from $t$ to $x_4/c$ on a real four-dimensional Euclidean manifold. It is not a mathematical operation applied to physics; it is physics.

Every standard account of the canonical structure of quantum mechanics — Born 1926 [11], Schrödinger 1926 [10], Heisenberg 1925, Dirac 1928 [12], von Neumann 1932 [14], Stone–von Neumann [13, 14], Gleason 1957 [15], Hestenes 1966 onward [16], Adler 1995 [17] — has likewise been a description of consequences of $dx_4/dt = ic$ articulated in formal languages that did not identify the physical fact. The Schrödinger equation, the canonical commutator, the Born rule, the Dirac equation, the spin-$\frac{1}{2}$ structure, the matter-antimatter pairing: all are theorems of the McGucken Principle [30, 33, 43, 29, 32, 31].

Every standard account of the symmetries and conservation laws of physics — Lorentz 1904, Poincaré 1905, Noether 1918, Weyl 1929 [21], Yang–Mills 1954, Coleman–Mandula 1967, Lüders–Pauli on CPT — has been a description of consequences of $dx_4/dt = ic$. The Lorentz group, the Poincaré group, the gauge groups, the conservation laws, the CPT theorem: all descend from the McGucken Symmetry [27, 35].

Every calculation in modern physics that uses an $i$ — in lattice QCD [44], in finite-temperature field theory [3, 4], in Euclidean quantum gravity [5, 36], in instanton physics, in Hawking radiation [6, 36], in scattering amplitudes via the amplituhedron [42], in twistor calculations [24, 41], in standard quantum-mechanical evolution [10, 30], in Dirac-equation solutions [12, 29], in any gauge-theoretic phase [21, 53] — is a calculation invoking the McGucken Principle [56, 27, 35], regardless of whether the physicist performing the calculation knows it. The calculation produces correct results either way. But the physicist who knows it has access to a unification the standard framework does not provide: every $i$ in physics is a record of the same physical fact, namely that the fourth dimension is expanding at the velocity of light.

The thirty-four theorems of this paper, combined with the broader McGucken corpus [38, 36, 30, 37, 39], establish that the McGucken Principle is the source-relation of which every standard input of modern theoretical physics is a downstream theorem. As the Wick rotation, the canonical structure of quantum mechanics, the symmetries and conservation laws of physics, the foundational atom of spacetime, and the Dirac equation are all completed along structurally-independent routes from the same foundational physical invariant $dx_4/dt = ic$, $dx_4/dt = ic$ must thusly represent a foundational mathematical and physical proof. The structural unification is matched by empirical confirmation: the McGucken Cosmology takes first place against every dark-sector and modified-gravity framework across twelve independent observational tests — the SPARC radial acceleration relation, Pantheon+ Type Ia supernovae, DESI 2024 baryon acoustic oscillations, the redshift-space-distortion growth rate $f\sigma_8(z)$, cosmic chronometer $H(z)$, the SPARC baryonic Tully-Fisher relation slope, the dark-energy equation of state, the $H_0$ tension, the Bullet Cluster lensing-versus-gas spatial offset, the dwarf-galaxy radial acceleration relation universality, and the extended baryonic Tully-Fisher relation across four decades of mass — and accomplishes this with zero free dark-sector parameters [40]. Empirical first place across the cosmological observational record is the experimental signature of the same physical principle from which the Wick rotation, every $i$ in physics, and the foundational structure of quantum theory, gravity, and thermodynamics descend as theorems.

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| **Framework** | **Extra dimension(s)** | **Where the $i$ appears (and what it is)** |
|---|---|---|
| Kaluza–Klein (1921, 1926) | Compactified fifth dimension at $\ell_P$ | U(1) gauge phase $e^{iq\alpha(x)}$ — McGucken $i$ via mechanism (c) |
| String theory (1970s–) | Six Calabi–Yau-wrapped dimensions at $\ell_P$ | Worldsheet complex structure $z = \sigma_1 + i\sigma_2$, holomorphic factorization, modular invariance — McGucken $i$ via mechanism (b) |
| M-theory (Witten 1995) | Eleventh dimension decompactifying at strong coupling | Modular structure of one-loop amplitudes, complex moduli-space structure, holomorphic BPS structure — McGucken $i$ via mechanisms (b), (c) |
| AdS/CFT (1997–98) | Radial coordinate $z \in (0, \infty)$ | Bulk and boundary partition functions $e^{iS}$, dimension-mass relation, Ryu–Takayanagi area law — McGucken $i$ via mechanisms (a), (b), (c) |
| **McGucken Principle** | **One dimension: $x_4$, advancing at rate $ic$** | **One $i$: the algebraic perpendicularity-marker of $x_4$ in $dx_4/dt = ic$** |

Table 1: The four extra-dimensional frameworks of theoretical physics and the imaginary structures they each carry. Under the McGucken Principle, all four frameworks describe the same fourth dimension $x_4$ advancing at $ic$, and all of their imaginary structures are projections of the same McGucken $i$.