The McGucken Sphere as Spacetime's Foundational Atom: A Complete Constructive Derivation of Twistor Space, the Positive Grassmannian, and the Amplituhedron from $dx_4/dt = ic$

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1 Abstract

This paper gives a single constructive derivation of the thesis that the McGucken sphere is the foundational atom of spacetime. The McGucken Principle,

$$\frac{dx_4}{dt} = ic,$$

is taken as the physical postulate that the fourth dimension expands spherically at the invariant speed of light, with i marking perpendicularity to the three ordinary spatial dimensions. The elementary object generated by this postulate is the McGucken sphere: the expanding null wavefront centered on an event. In this sense, the McGucken sphere functions as the foundational atom of spacetime, because it is the primitive null-incidence unit from which metric structure, causal propagation, twistor incidence, quantum phase-flow, and scattering geometry are generated. From this atom, the paper derives Penrose incidence $\omega^A = ix^{AA'}\pi_{A'}$, projective twistor space CP^3, planar momentum twistors $Z_a = (\lambda_a, x_a\lambda_a)$, McGucken-positive external data $M_+(k+4, n)$, positive Grassmannian boundary measurements $C_{\alpha a}$, Huygens superposition $Y = CZ$, canonical $d\log$ forms, BCFW cells, positroid stratification, loop data $D_{(i)}$, and the full loop amplituhedron $G_+(k, n; L)$. The paper also fixes the exact convention match between Witten's twistor-string curve degree $d = q - 1 + \ell$ and the amplituhedron convention $q = k_{\text{A}} + 2$, yielding $d = k_{\text{A}} + 1 + \ell$. The resulting chain is

$$dx_4/dt = ic \Rightarrow \Sigma_+(p) \Rightarrow CP^3 \Rightarrow Z_a \Rightarrow M_+(k+4, n) \Rightarrow G_+(k, n) \Rightarrow G_+(k, n; L) \Rightarrow Y = CZ, \quad \mathcal{L}_{(i)} = D_{(i)}Z \Rightarrow$$

The amplituhedron is thereby interpreted not merely as an abstract positive geometry, but as the positive-geometric image of x_4-generated Huygens phase-flow networks built from the McGucken sphere as the foundational atom of spacetime.

2 Status of Claims

The amplituhedron is an established construction for planar $N = 4$ super-Yang-Mills amplitudes, defined using positive external data Z_a, positive matrices $C \in G_+(k, n)$, and the map $Y = CZ$ into $G(k, k+4)$ [5]. The positive Grassmannian program relates planar on-shell diagrams to positive Grassmannian cells, positive coordinates, invariant $d\log$ measures, BCFW bridges, decorated permutations, and Yangian invariance [6]. Positive geometries have canonical forms characterized by logarithmic singularities on boundaries and residues equal to canonical forms of those boundaries [7]. Momentum twistors encode planar null-polygon kinematics through region momenta $p_i = x_i - x_{i-1}$ and twistor variables Z_i^α [8].

The McGucken Principle is a proposed physical postulate and is treated here conditionally: if $dx_4/dt = ic$ is accepted as the underlying geometric principle, the following theorems show how the twistor and amplituhedron structures arise from it. The physical postulate base comes from the McGucken quantum-mechanics paper, the McGucken Feynman-diagram paper, the McGucken Penrose-twistor paper, and the McGucken Witten-twistor-programme paper [1–4].

3 Introduction

The central proposal of this paper is that the McGucken sphere is the foundational atom of spacetime. An atom, in this usage, is not a material particle but an indivisible generative unit: the smallest geometric act from which the structures of spacetime, quantum propagation, twistor incidence, and scattering geometry may be reconstructed. The McGucken sphere is this unit because $dx_4/dt = ic$ assigns to every event an expanding fourth-dimensional null wavefront, whose projection into ordinary spacetime is the future light cone.

On this view, spacetime is not first a passive manifold on which light cones are later drawn. Rather, each event generates a McGucken sphere, and the network of these expanding spheres supplies the primitive incidence relations out of which spacetime geometry is built. The metric records the algebra of this expansion. Causality records the nesting and intersection of these spheres. Quantum propagation records the coherent phase-flow along their null generators. Twistor space records their projectivized null-generator geometry. The positive Grassmannian records positive boundary measurements of their allowed intersection networks. The amplituhedron records the final positive-geometric image of those networks.

Thus the constructive thesis is:

$$\boxed{\text{McGucken sphere} = \text{foundational atom of spacetime}}$$

and the derivational programme is:

$$\boxed{\text{foundational atom} \to \text{null incidence} \to \text{twistors} \to \text{positive Grassmannian} \to \text{amplituhedron}}.$$

The paper is organized as a chain of formal definitions, theorems, proofs, and plain-language explanations. Each stage shows how a standard structure in quantum mechanics, twistor theory, scattering amplitudes, or positive geometry can be reconstructed from the McGucken sphere once $dx_4/dt = ic$ is adopted as the underlying geometric postulate.

4 Foundational Postulate

4.1 Postulate 1: The McGucken Principle

The fourth dimension expands spherically at the invariant speed of light:

$$\frac{dx_4}{dt} = ic.$$

Here $x_4 = ict$, and the imaginary unit i records the perpendicularity of the fourth coordinate to the ordinary spatial coordinates x_1, x_2, x_3. The McGucken quantum-mechanics paper treats this postulate as the origin of complex quantum phase, the wavefunction, the Born rule, nonlocality, and the Feynman apparatus [1].

4.2 Theorem 1: $x_4 = ict$ Generates the Minkowski Metric

The substitution $x_4 = ict$ transforms the Euclidean four-coordinate line element into the Minkowski line element.

4.2.1 Proof

Begin with the four-coordinate expression

$$ds_E^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2.$$

Using $x_4 = ict$,

$$dx_4^2 = (icdt)^2 = -c^2dt^2.$$

Therefore

$$ds_E^2 = dx_1^2 + dx_2^2 + dx_3^2 - c^2 dt^2.$$

This is the Minkowski interval in mostly-plus convention. Hence the Lorentzian signature is generated by the perpendicular fourth-dimensional coordinate $x_4 = ict$. □

4.3 Theorem 2: The McGucken Sphere Is the Future Null Cone and the Foundational Atom of Spacetime

The spherical expansion of x_4 at speed c projects into ordinary spacetime as the future null cone

$$\Sigma_+(p) = \{x : (x-p)^2 = 0,\ x^0 > p^0\}.$$

This null sphere is the foundational atom of spacetime: the primitive causal-incidence unit from which local metric structure, null propagation, twistor incidence, and positive scattering geometry are successively generated.

4.3.1 Proof

Set $ds^2 = 0$ in the line element derived above:

$$dx_1^2 + dx_2^2 + dx_3^2 - c^2 dt^2 = 0.$$

Therefore

$$|\mathbf{x} - \mathbf{x}_0|^2 = c^2 (t - t_0)^2.$$

For $t > t_0$, this is a spatial sphere of radius $c(t - t_0)$. In spacetime, the union of these expanding spheres is the future null cone of $p = (t_0, \mathbf{x}_0)$. This is the McGucken Sphere $\Sigma_+(p)$. Because $\Sigma_+(p)$ is generated at each event and supplies the primitive null-incidence relation for all later constructions, it is the foundational atom of spacetime in the present theory. □

4.4 Plain-language meaning

The McGucken Sphere is simply the light cone seen as a real expanding sphere produced by the fourth dimension. Instead of treating the light cone as a kinematic constraint, this framework treats it as the projection of x_4-expansion. This is why the McGucken sphere is called the foundational atom of spacetime: every event contributes one such primitive expanding null unit, and the relations among these units generate causal, quantum, twistor, and amplituhedral structure.

5 Complex Phase and Quantum Amplitude

5.1 Theorem 3: The Rest-Frame Quantum Phase Follows from x_4-Evolution

The rest-frame phase

$$\psi(t) = \psi(0) \exp\left(-\frac{imc^2 t}{\hbar}\right)$$

is the unitary phase associated with fourth-dimensional evolution.

5.1.1 Proof

The rest energy is

$$E_0 = mc^2.$$

By the Planck relation,

$$E_0 = \hbar\omega_C,$$

so

$$\omega_C = \frac{mc^2}{\hbar}.$$

Unitary evolution generated by this frequency gives

$$\psi(t) = \psi(0)e^{-i\omega_C t}.$$

Substitution yields

$$\psi(t) = \psi(0)\exp\left(-\frac{imc^2t}{\hbar}\right).$$

In the McGucken interpretation, the i in this quantum phase is the same perpendicularity marker as in $x_4 = ict$. □

5.2 Theorem 4: The Path Integral Is Iterated Huygens Propagation

The path integral

$$K(b,a) = \int \mathcal{D}[x]\exp\left(\frac{iS[x]}{\hbar}\right)$$

is obtained by composing infinitesimal Huygens propagation kernels generated by the expanding fourth-dimensional wavefront.

5.2.1 Proof

Divide a time interval into N steps of width ϵ. The propagator is composed as

$$K(b,a) = \lim_{N\to\infty}\int \prod_{j=1}^{N-1} dx_j \prod_{j=0}^{N-1} K_\epsilon(x_{j+1}, x_j).$$

For a short interval,

$$K_\epsilon(x_{j+1}, x_j) = \left(\frac{m}{2\pi i\hbar\epsilon}\right)^{1/2}\exp\left[\frac{i}{\hbar}\epsilon L\left(x_j, \frac{x_{j+1}-x_j}{\epsilon}\right)\right].$$

Multiplying the exponential factors gives

$$\exp\left[\frac{i}{\hbar}\sum_j \epsilon L_j\right].$$

Taking the limit gives

$$\sum_j \epsilon L_j \to S[x],$$

and therefore

$$K(b,a) = \int \mathcal{D}[x] \exp\left(\frac{iS[x]}{\hbar}\right).$$

The McGucken interpretation identifies each infinitesimal kernel with coherent Huygens propagation on an x_4-generated wavefront. □

6 Feynman Diagrams from Huygens-with-Interaction

6.1 Theorem 5: The Dyson Expansion Is Iterated Huygens-with-Interaction

The perturbative S-matrix

$$S = T \exp\left[-\frac{i}{\hbar}\int dt\, H_I(t)\right]$$

is the operator expression of iterated Huygens propagation with localized interaction insertions.

6.1.1 Proof

Expand the time-ordered exponential:

$$S = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{i}{\hbar}\right)^n \int dt_1 \cdots dt_n\, T\{H_I(t_1)\cdots H_I(t_n)\}.$$

Writing

$$H_I(t) = \int d^3x\, \mathcal{H}_I(x,t),$$

each term becomes an integral over n spacetime interaction points. Wick contraction of the time-ordered products supplies propagators between those points. Thus every term is a sum over graphs made of vertices, propagators, external lines, and loop integrations.

In the McGucken interpretation, propagators are x_4-coherent Huygens kernels, vertices are local x_4-phase-exchange loci, and loops are closed internal x_4-phase chains. Therefore Feynman diagrams are combinatorial representations of iterated McGucken-sphere interactions. □

6.2 Proposition 1: Propagator as x_4-Coherent Green Function

The Feynman propagator

$$D_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik\cdot(x-y)}$$

is an x_4-coherent Green function with the $i\epsilon$ prescription selecting the forward x_4-orientation.

6.2.1 Proof

The Klein-Gordon Green function in momentum space satisfies

$$(-k^2 + m^2)\tilde{D}(k) = -i.$$

Thus

$$\tilde{D}(k) = \frac{i}{k^2 - m^2}.$$

The poles at $k^2 = m^2$ require a contour prescription. The Feynman prescription gives

$$\tilde{D}_F(k) = \frac{i}{k^2 - m^2 + i\epsilon}.$$

The $+i\epsilon$ prescription selects the causal contour. In the McGucken interpretation, this contour is the algebraic expression of the forward x_4-orientation. □

7 Penrose Twistor Space from McGucken Incidence

7.1 Definition 1: McGucken Twistor

A McGucken twistor is a projective spinor pair

$$Z^\alpha = (\omega^A, \pi_{A'})$$

with incidence relation

$$\omega^A = ix^{AA'}\pi_{A'}.$$

The McGucken twistor paper identifies this relation as the physical origin of Penrose twistor incidence, with i inherited from $x_4 = ict$ [3].

7.2 Theorem 6: McGucken Sphere Incidence Generates CP^3

For each spacetime point x, the null directions of the McGucken Sphere define a CP^1 line in projective twistor space; the union of these incidence lines generates CP^3.

7.2.1 Proof

At fixed x, a null direction is represented by a nonzero spinor $\pi_{A'}$, modulo projective rescaling:

$$\pi_{A'} \sim r\pi_{A'}, \qquad r \in \mathbb{C}^*.$$

The space of such directions is CP^1. For each $\pi_{A'}$, define

$$\omega^A = ix^{AA'}\pi_{A'}.$$

Then

$$Z^\alpha = (\omega^A, \pi_{A'})$$

is a point of CP^3, and the set of all such Z^α for fixed x forms a projective line CP^1. Varying x sweeps out projective twistor space. Therefore CP^3 is the projectivized incidence geometry of McGucken null spheres. □

7.3 Theorem 7: Null Rays Correspond to Twistor Points

A null generator of a McGucken Sphere corresponds to a point in projective twistor space.

7.3.1 Proof

A null generator is specified by x and projective spinor $\pi_{A'}$. Incidence gives

$$Z^\alpha = (ix^{AA'}\pi_{A'}, \pi_{A'}).$$

Rescaling $\pi_{A'} \mapsto r\pi_{A'}$ rescales $Z^\alpha \mapsto rZ^\alpha$, leaving the projective twistor point unchanged. Therefore each null generator defines one projective twistor point. □

7.4 Plain-language meaning

An event emits a sphere of null directions. Projectivize those directions and the event becomes a CP^1 line in twistor space. This gives a direct physical reading of Penrose's point-line correspondence.

8 Planar Momentum Twistors

8.1 Definition 2: External McGucken Null Polygon

An external planar scattering process is represented by region momenta

$$x_0, x_1, \ldots, x_n = x_0$$

with null edges

$$p_a = x_a - x_{a-1}, \qquad p_a^2 = 0.$$

Each edge is a null generator of a McGucken Sphere and factorizes as

$$p_a^{AA'} = \lambda_a^A \tilde{\lambda}_a^{A'}.$$

8.2 Definition 3: Momentum Twistor

Given the null polygon, define

$$Z_a = (\lambda_a, \mu_a), \qquad \mu_a = x_a\lambda_a.$$

Hodges' momentum twistors encode planar region-momentum kinematics, with $p_i = x_i - x_{i-1}$ and null polygon edges represented in twistor variables [8].

8.3 Theorem 8: Momentum Twistors Are Planar McGucken Incidence Data

The momentum twistor relation

$$Z_a = (\lambda_a, x_a\lambda_a)$$

is the planar null-polygon specialization of McGucken incidence

$$\omega^A = ix^{AA'}\pi_{A'}.$$

8.3.1 Proof

McGucken incidence maps a spinor direction and spacetime point to a twistor:

$$(\pi_{A'}, x) \mapsto (ix^{AA'}\pi_{A'}, \pi_{A'}).$$

Momentum twistor incidence maps a planar region point and spinor direction to

$$(\lambda_a, x_a) \mapsto (\lambda_a, x_a\lambda_a).$$

Both constructions multiply a null spinor direction by the appropriate spacetime or region-spacetime coordinate to form the second twistor component. The explicit i in the McGucken incidence is absorbed into the standard complex twistor convention used for momentum twistors. Therefore momentum twistors are the planar region-momentum version of McGucken twistor incidence. □

9 Positive External Data

9.1 Definition 4: McGucken-Positive External Configuration

A McGucken-positive external configuration is a cyclically ordered set of projective momentum twistors

$$Z_1, \dots, Z_n \in \mathbb{P}^{k+3}$$

such that every ordered $(k+4)$-minor is positive:

$$\langle Z_{a_1} \cdots Z_{a_{k+4}} \rangle > 0 \quad \text{for} \quad a_1 < \cdots < a_{k+4}.$$

9.2 Theorem 9: Ordered x_4-Phase Gives Positive External Data

Let

$$\theta_1 < \theta_2 < \cdots < \theta_n$$

be ordered x_4-phase parameters and define $t_a = e^{\theta_a}$. The moment-curve representative

$$Z_a = (1, t_a, t_a^2, \dots, t_a^{k+3})$$

has all ordered maximal minors positive.

9.2.1 Proof

For any ordered subset $a_1 < \cdots < a_{k+4}$, the determinant is Vandermonde:

$$\det(t_{a_j}^{i-1})_{i,j=1}^{k+4} = \prod_{1 \le r < s \le k+4} (t_{a_s} - t_{a_r}).$$

Since $t_{a_1} < \cdots < t_{a_{k+4}}$, every factor is positive. Therefore every ordered maximal minor is positive. □

9.3 Theorem 10: Classification of McGucken-Positive External Data

The moduli space of McGucken-positive external configurations is

$$M_+(k+4,n)/GL^+(k+4),$$

where $M_+(k+4,n)$ is the space of $(k+4)\times n$ matrices with all ordered maximal minors positive.

9.3.1 Proof

Choose homogeneous representatives for the n projective twistors and place them as columns of Z. Positivity of all ordered maximal minors means $Z \in M_+(k+4,n)$. Multiplication by $G \in GL^+(k+4)$ multiplies all maximal minors by $\det G > 0$, preserving positivity. Positive column rescalings also preserve positivity. Hence each McGucken-positive configuration determines a point of $M_+(k+4,n)/GL^+(k+4)$.

Conversely, any $Z \in M_+(k+4,n)$ defines projective twistors with positive ordered minors. These are precisely McGucken-positive external data. □

10 Witten Twistor-Curve Localization

10.1 Theorem 11: Common McGucken Origin Gives Holomorphic Twistor Support

If massless external states arise from x_4-stationary null generators of a common McGucken sphere-intersection process, then their twistor representatives lie on holomorphic support determined by the number of independent coherent x_4-phase channels.

10.1.1 Proof

Massless McGucken propagation follows null generators, and null generators correspond to twistor points by Theorem 7. A single common emission origin gives a CP^1 line of null directions in twistor space. Adding independent interaction channels glues additional projective null-direction components. Algebraically, this increases the degree of the holomorphic support. This matches the Witten twistor-programme pattern described in the McGucken Witten paper: MHV amplitudes localize on lines, NMHV amplitudes on conics, and higher sectors on higher-degree curves [4]. □

10.2 Theorem 12: Exact Degree Convention

Let q be the number of negative-helicity gluons, ℓ the loop order, and k_{A} the amplituhedron convention with MHV $k_{\mathrm{A}} = 0$. Then the Witten twistor-curve degree is

$$d = q - 1 + \ell = k_{\mathrm{A}} + 1 + \ell.$$

10.2.1 Proof

Witten's twistor-string formula states

$$d = q - 1 + \ell,$$

where q is the number of negative-helicity gluons and ℓ is loop order [9]. In the amplituhedron convention, MHV is $k_{\mathrm{A}} = 0$, NMHV is $k_{\mathrm{A}} = 1$, and $M_{n,k}$ has Grassmann weight $4(k+2)$ [5], so

$$q = k_A + 2$$

Substitution gives

$$d = (k_A + 2) - 1 + \ell = k_A + 1 + \ell.$$

□

Sector	k_A	q	**Tree degree** d	**Support**
MHV	0	2	1	Line
NMHV	1	3	2	Conic
N^2MHV	2	4	3	Cubic
N^rMHV	r	$r+2$	$r+1$	Degree $r+1$ curve
L-loop N^rMHV	r	$r+2$	$r+1+L$	Degree $r+1+L$, genus $\leq L$

11 Positive Grassmannian from McGucken Networks

11.1 Definition 5: McGucken Intersection Network

A McGucken intersection network is a directed planar graph with:

1. n ordered boundary leaves.
2. k independent source channels.
3. Internal vertices representing McGucken-sphere intersections.
4. Positive edge weights

$$\alpha_e = e^{\rho_e} > 0,$$

where ρ_e is an additive x_4-flux coordinate.

11.2 Definition 6: Boundary Measurement Matrix

Define

$$C_{\alpha a} = \sum_{\gamma:\alpha \to a} \prod_{e \in \gamma} \alpha_e,$$

where γ runs over directed paths from source α to boundary leaf a.

11.3 Theorem 13: McGucken Networks Define $G_+(k, n)$

For a planar directed McGucken network with positive edge weights and compatible boundary orientation, the boundary measurement matrix C lies in $G_+(k, n)$ on the corresponding positroid cell.

11.3.1 Proof

Consider an ordered minor

$$\Delta_A(C) = \det(C_{\alpha a_\beta}).$$

Expanding by multilinearity gives a signed sum over path families:

$$\Delta_A(C) = \sum_{\gamma_1,\ldots,\gamma_k} \mathrm{sgn}(\gamma_1,\ldots,\gamma_k) \prod_{\beta=1}^{k} \prod_{e\in\gamma_\beta} \alpha_e.$$

For planar directed networks with compatible orientation, intersecting path families cancel or do not contribute to the reduced boundary measurement determinant. The surviving terms are nonintersecting path families with positive sign. Therefore

$$\Delta_A(C) = \sum_{\Gamma:\text{nonintersecting}} \prod_{e\in\Gamma} \alpha_e.$$

Since every $\alpha_e > 0$, each nonzero minor is positive. The pattern of positive and vanishing minors defines the corresponding positroid cell. This is the standard boundary-measurement relation between planar directed networks and the totally nonnegative Grassmannian [10]. □

12 BCFW Bridges and Positroid Cells

12.1 Definition 7: McGucken BCFW Bridge

A McGucken BCFW bridge between adjacent boundary legs a and $a+1$ inserts a positive x_4-flux channel with

$$\alpha = e^{\rho} > 0$$

and transforms the boundary measurement matrix by

$$c_{a+1} \mapsto c_{a+1} + (-1)^q \alpha c_a.$$

It contributes the measure factor

$$\frac{d\alpha}{\alpha}.$$

The positive Grassmannian construction identifies BCFW bridges with adjacent transpositions, canonical positive coordinates, and $d\log\alpha$ measures [6].

12.2 Theorem 14: Every Tree BCFW Cell Arises from a Reduced McGucken Network

Every BCFW cell used in the tree amplituhedron can be generated from the identity network by a sequence of McGucken BCFW bridges.

12.2.1 Proof

The positive Grassmannian construction states that reduced planar diagrams and decorated permutations can be built from the identity by bridge decompositions, with each bridge acting by an adjacent transposition and introducing a positive coordinate α [6]. A McGucken BCFW bridge is defined as the same adjacent-transposition operation, interpreted as an elementary positive x_4-flux channel. Therefore the bridge sequence generating any BCFW cell also generates a reduced McGucken network with the same $C(\alpha)$. □

12.3 Theorem 15: Every Reduced McGucken Network Defines an Allowed Positroid Cell

Every reduced McGucken network determines a positroid cell of $G_+(k,n)$, and equivalent reduced networks determine the same cell.

12.3.1 Proof

By Theorem 13, the boundary measurement matrix has nonnegative ordered minors and a definite pattern of positive and vanishing Plücker coordinates. This pattern is a positroid. The positive Grassmannian literature labels such cells by decorated permutations and identifies reduced planar diagrams related by mergers, square moves, and bubble deletion as equivalent representatives of the same on-shell form [6]. Hence every reduced McGucken network defines an allowed positroid cell, and equivalent networks define the same cell. □

13 Huygens Superposition and the Amplituhedron Map

13.1 Theorem 16: Huygens Superposition Gives $Y = CZ$

Let Z_a be positive external momentum-twistor data and $C_{\alpha a}$ the McGucken boundary measurement matrix. The internal k-plane is

$$Y^I_\alpha = C_{\alpha a} Z^I_a.$$

Thus

$$Y = CZ.$$

13.1.1 Proof

Huygens propagation is linear at the level of amplitudes. The total twistor in internal channel α is the coherent sum of all external twistors weighted by the total boundary measurement from α to a:

$$Y_\alpha = \sum_a C_{\alpha a} Z_a.$$

In components,

$$Y^I_\alpha = C_{\alpha a} Z^I_a.$$

This is exactly the amplituhedron map $Y = CZ$ [5]. □

14 Canonical $d\log$ Forms

14.1 Theorem 17: x_4-Flux Coordinates Generate $d\log$ Forms

If

$$\alpha_i = e^{\rho_i} > 0,$$

then the translation-invariant x_4-flux measure becomes

$$d\rho_i = \frac{d\alpha_i}{\alpha_i}.$$

Therefore a cell with independent flux coordinates has form

$$\Omega_\Gamma = \prod_i \frac{d\alpha_i}{\alpha_i}.$$

14.1.1 Proof

Differentiate $\alpha_i = e^{\rho_i}$:

$$d\alpha_i = e^{\rho_i} d\rho_i = \alpha_i d\rho_i.$$

Thus

$$d\rho_i = \frac{d\alpha_i}{\alpha_i}.$$

Taking the product over independent coordinates gives

$$\prod_i d\rho_i = \prod_i \frac{d\alpha_i}{\alpha_i}.$$

The amplituhedron cell form is precisely a product of $d\alpha_i/\alpha_i$ over positive coordinates [5]. □

14.2 Theorem 18: Pushforward Gives the Canonical Form

Let

$$\Phi : \{\alpha_i > 0\} \to Y = C(\alpha)Z$$

be an orientation-preserving parametrization of an amplituhedron cell. Then

$$\Phi_* \left(\prod_i \frac{d\alpha_i}{\alpha_i} \right)$$

is the canonical form on that cell.

14.2.1 Proof

Positive geometry theory defines canonical forms by logarithmic singularities on boundaries and residues equal to canonical forms on those boundaries. It also states that orientation-preserving pushforwards of canonical forms give canonical forms of image geometries [7]. The positive coordinate domain has canonical form $\prod_i d\alpha_i/\alpha_i$. Therefore the pushforward under $Y = C(\alpha)Z$ is the canonical form of the image cell. □

15 Boundary Stratification, Locality, and Unitarity

15.1 Theorem 19: Residues Are Boundary Network Forms

The residue of

$$\Omega = \frac{d\alpha_j}{\alpha_j} \wedge \prod_{i \neq j} \frac{d\alpha_i}{\alpha_i}$$

at $\alpha_j = 0$ is the canonical form of the boundary network.

15.1.1 Proof

Taking the residue gives

$$\mathrm{Res}_{\alpha_j=0}\, \Omega = \prod_{i \neq j} \frac{d\alpha_i}{\alpha_i}.$$

The remaining variables parametrize the network with the j-th channel deleted or contracted. This is the boundary network. Positive-geometry canonical forms are recursively characterized by exactly this residue property [7]. □

15.2 Theorem 20: Locality Boundaries Are Null McGucken-Sphere Separations

The physical pole

$$\langle Y_1 \cdots Y_k Z_i Z_{i+1} Z_j Z_{j+1} \rangle = 0$$

is the twistor expression of a null separation between region points x_i and x_j.

15.2.1 Proof

In momentum-twistor geometry, adjacent pairs (Z_i, Z_{i+1}) encode the region point x_i, and brackets involving $(Z_i, Z_{i+1}, Z_j, Z_{j+1})$ encode region-space separations. Hodges' construction relates such twistor brackets to region-momentum invariants [8]. In McGucken geometry, null separation means two region points are connected by a shared null sphere boundary. Therefore the amplituhedron locality boundary is the momentum-twistor image of a McGucken null-sphere separation. □

15.3 Theorem 21: Unitarity Cuts Open Closed x_4-Chains

Loop unitarity cuts correspond to opening closed internal x_4-phase chains into on-shell boundary channels.

15.3.1 Proof

In perturbative QFT, a unitarity cut places internal propagators on shell and replaces a loop contribution by products of lower on-shell amplitudes. In the McGucken interpretation, loops are closed x_4-phase chains. Cutting the loop replaces the closure condition by two on-shell endpoints carrying matched phase-flow data. Hence unitarity cuts are opened closed x_4-chains. □

16 Loop Amplituhedron and $G_+(k, n; L)$

16.1 Definition 8: Closed x_4-Chain Boundary Measurement

For loop i, cut the closed x_4-chain into two boundary channels A_i and B_i, and define

$$D_{(i),1a} = \sum_{\gamma:A_i\to a} \prod_{e\in\gamma} \alpha_e, \qquad D_{(i),2a} = \sum_{\gamma:B_i\to a} \prod_{e\in\gamma} \alpha_e.$$

Together,

$$D_{(i)} = \begin{pmatrix} D_{(i),1a} \\ D_{(i),2a} \end{pmatrix}.$$

16.2 Theorem 22: McGucken Loop Positivity Equals $G_+(k, n; L)$

The McGucken loop-positive space equals the loop positive space $G_+(k, n; L)$.

16.2.1 Proof

The loop amplituhedron uses a k-plane C and L two-planes $D_{(i)}$ in the complement of C, with positivity of all ordered stacked minors of

$$\begin{pmatrix} D_{(i_1)} \\ \vdots \\ D_{(i_l)} \\ C \end{pmatrix}$$

for all subsets of loop indices [5]. The McGucken construction defines exactly the same algebraic data and inequalities, with $D_{(i)}$ interpreted as boundary measurements of cut-open closed x_4-chains. Hence the spaces coincide. $\square$

16.3 Theorem 23: Full Loop Amplituhedron Map

The full McGucken loop map is

$$Y^I_\alpha = C_{\alpha a} Z^I_a, \qquad \mathcal{L}^I_{(i),\gamma} = D_{(i),\gamma a} Z^I_a.$$

16.3.1 Proof

The full amplituhedron map sends tree data by $Y = CZ$ and loop data by $\mathcal{L}_{(i)} = D_{(i)} Z$ [5]. In the McGucken picture, the rows of C are open tree-channel boundary measurements, while the rows of $D_{(i)}$ are cut-open loop-chain boundary measurements. Huygens superposition applies to both, giving the stated map. $\square$

17 Yangian Invariance

17.1 Theorem 24: Yangian Invariance from Dual McGucken Conformal Symmetry

If the McGucken null-sphere construction is conformally invariant and the planar region-momentum null polygon inherits dual conformal invariance, then the induced positive-Grassmannian form is Yangian invariant.

17.1.1 Proof

Ordinary conformal transformations preserve null cones and therefore preserve McGucken sphere incidence. Dual conformal transformations act on the region-momentum polygon $x_0, \ldots, x_n$, whose edges are null momenta. Momentum twistors are designed to make this dual conformal structure natural [8].

The McGucken-to-Grassmannian map uses only incidence, cyclic order, positive path weights, and projective superposition:

$$Z_a = (\lambda_a, x_a\lambda_a), \qquad C_{\alpha a} = \sum_{\gamma:\alpha\to a} \prod_{e\in\gamma} \alpha_e, \qquad Y = CZ.$$

The $d\log$ measure is invariant under positive multiplicative rescalings of the flux variables. The positive Grassmannian literature identifies Yangian invariance with diffeomorphisms of $G(k, n)$ preserving the positive structure [6]. Therefore the induced McGucken positive-Grassmannian form carries the Yangian invariance generated by ordinary plus dual conformal symmetry. □

18 Operator-Algebraic Microcausality from McGucken Sphere Causality

One remaining structural problem is to translate the McGucken-sphere account of causal propagation into the operator-algebraic language of local quantum physics. In standard algebraic quantum field theory, local observables are assigned to spacetime regions by a net of algebras, and microcausality is encoded by the commutativity of algebras assigned to spacelike-separated regions [11]. In the Wightman formulation, microcausality appears as local commutativity or graded local commutativity of smeared fields with spacelike-separated supports [12].

The McGucken construction suggests a geometric origin for this algebraic condition. If the McGucken sphere is the foundational atom of spacetime, then causal influence is carried only along the null incidence structure generated by these spheres. Two regions whose McGucken causal completions do not intersect cannot exchange x_4-Huygens phase-flow. Therefore the algebras generated by observables localized in those regions must commute, or graded-commute, when embedded into the quasilocal algebra.

18.1 Definition 9: McGucken Causal Completion

For an open bounded spacetime region O, define its McGucken causal completion by

$$O_M^\diamond = \bigcup_{p\in O} \left(\Sigma_+(p) \cup \Sigma_-(p)\right),$$

where $\Sigma_+(p)$ and $\Sigma_-(p)$ are the future and past McGucken spheres centered at p. Equivalently, $O_M^\diamond$ is the smallest region containing O and all events connected to O by McGucken null-sphere incidence.

18.2 Definition 10: McGucken Local Net

A McGucken local net is an assignment

$$O \longmapsto \mathcal{A}_M(O)$$

from bounded open spacetime regions to unital $*$-algebras, C^*-algebras, or von Neumann algebras on a common Hilbert space $\mathcal{H}$, satisfying:

1. Isotony:

$$O_1 \subset O_2 \implies \mathcal{A}_M(O_1) \subset \mathcal{A}_M(O_2).$$

1. McGucken causal covariance:

$$g\Sigma(p) = \Sigma(gp) \quad \Longrightarrow \quad U(g)\mathcal{A}_M(O)U(g)^{-1} = \mathcal{A}_M(gO),$$

for every transformation g preserving McGucken null-sphere incidence.

1. McGucken causal locality:

$$O_1{}^{\diamond}_M \cap O_2{}^{\diamond}_M = \varnothing \quad \Longrightarrow \quad [\mathcal{A}_M(O_1), \mathcal{A}_M(O_2)]_{\text{gr}} = 0.$$

Here $[\cdot,\cdot]_{\text{gr}}$ is the graded commutator, reducing to the ordinary commutator for bosonic observables.

18.3 Theorem 26: McGucken Causal Locality Implies Algebraic Microcausality

Let O_1 and O_2 be bounded open regions. Suppose no McGucken sphere generated from O_1 intersects O_2, and no McGucken sphere generated from O_2 intersects O_1. Then the associated local algebras commute, or graded-commute:

$$[\mathcal{A}_M(O_1), \mathcal{A}_M(O_2)]_{\text{gr}} = 0.$$

18.3.1 Proof

By hypothesis,

$$O_1{}^{\diamond}_M \cap O_2{}^{\diamond}_M = \varnothing.$$

The McGucken sphere is the primitive carrier of causal incidence. Therefore an observable localized in O_1 can influence another observable only through an x_4-phase-flow chain lying in $O_1{}^{\diamond}_M$. Similarly, an observable localized in O_2 can influence another observable only through a chain lying in $O_2{}^{\diamond}_M$. Since the two causal completions are disjoint, there exists no McGucken incidence chain joining the two regions.

Let $A \in \mathcal{A}_M(O_1)$ and $B \in \mathcal{A}_M(O_2)$. The operational content of A is exhausted by operations supported inside $O_1{}^{\diamond}_M$, and the operational content of B is exhausted by operations supported inside $O_2{}^{\diamond}_M$. Since these regions have no common McGucken incidence channel, their order of operation is unobservable. Hence

$$AB = (-1)^{|A||B|}BA,$$

which is precisely

$$[A, B]_{\text{gr}} = 0.$$

Since A and B were arbitrary, the algebras graded-commute. □

18.4 Corollary 6: Standard Spacelike Microcausality

If O_1 and O_2 are spacelike separated in the metric induced by $x_4 = ict$, then

$$[\mathcal{A}_M(O_1), \mathcal{A}_M(O_2)]_{\text{gr}} = 0.$$

18.4.1 Proof

By Theorem 1, $x_4 = ict$ induces the Minkowski interval. By Theorem 2, McGucken spheres are the null cones of that interval. Spacelike separation means no future or past null cone from O_1 reaches O_2, and no future or past null cone from O_2 reaches O_1. Therefore

$$O_{1M}^{\diamond} \cap O_{2M}^{\diamond} = \varnothing,$$

and Theorem 26 gives the result. □

18.5 Theorem 27: Smeared-Field Microcausality

Let $\Phi(f)$ and $\Psi(g)$ be smeared fields affiliated with $\mathcal{A}_M(O_f)$ and $\mathcal{A}_M(O_g)$, where

$$\mathrm{supp}(f) \subset O_f, \qquad \mathrm{supp}(g) \subset O_g.$$

If O_f and O_g are McGucken-spacelike separated, then

$$[\Phi(f), \Psi(g)]_{\mathrm{gr}} = 0.$$

18.5.1 Proof

Because $\Phi(f)$ is affiliated with $\mathcal{A}_M(O_f)$ and $\Psi(g)$ is affiliated with $\mathcal{A}_M(O_g)$, their commutator is governed by the commutator of the corresponding local algebras. McGucken-spacelike separation gives

$$O_{fM}^{\diamond} \cap O_{gM}^{\diamond} = \varnothing.$$

By Theorem 26,

$$[\mathcal{A}_M(O_f), \mathcal{A}_M(O_g)]_{\mathrm{gr}} = 0.$$

Therefore every pair of affiliated smeared fields graded-commutes:

$$[\Phi(f), \Psi(g)]_{\mathrm{gr}} = 0.$$

This is the Wightman local-commutativity condition expressed in McGucken-sphere language. □

18.6 Proposition 2: Causal Completion as Primitive Algebraic Localization

The natural localization region for a McGucken observable is not merely the coordinate support O, but its McGucken causal completion $O_M^{\diamond}$.

18.6.1 Proof

An observable localized in O can be prepared, propagated, and detected only through x_4-Huygens phase-flow generated by McGucken spheres centered in O. Hence its full operational support includes all null incidences accessible from O, namely $O_M^{\diamond}$. If two coordinate regions overlap only after causal completion, then their observables may fail to commute because the overlap represents a shared x_4-incidence channel. If their causal completions remain disjoint, no such channel exists, and microcausality follows by Theorem 26. □

18.7 Plain-Language Explanation

In ordinary quantum field theory, microcausality is often stated as a rule: operators at spacelike separation commute. The McGucken construction explains why this should be true. Operators fail to commute only when their corresponding physical operations can influence one another. But influence propagates through McGucken spheres. Therefore if two regions cannot be connected by any McGucken sphere or chain of McGucken-sphere intersections, their operators must commute.

The algebraic translation is therefore:

$$\text{no shared McGucken sphere incidence} \quad \Longrightarrow \quad \text{commuting local algebras.}$$

This turns microcausality from an added axiom into the operator-algebraic shadow of the McGucken sphere as the foundational atom of spacetime.

19 Toward a McGucken-Informed Gravitational Twistor String for Einstein Gravity

The second remaining structural problem is to extend the McGucken-twistor-amplituhedron chain from gauge-theoretic amplitudes to full Einstein gravity. Standard twistor-string theory began with gauge theory and conformal supergravity [9]. Subsequent work developed twistor-string or twistor-inspired formulae for Einstein supergravity tree amplitudes, including Adamo-Mason constructions, Cachazo-Skinner rational-curve formulae, Cachazo-Mason-Skinner Grassmannian formulations, Skinner's $N = 8$ twistor string, and ambitwistor-string formulations of gravitational scattering [13–18].

The McGucken Principle adds a physical interpretation to this line of work. Since the McGucken sphere is the foundational atom of spacetime, gravity should not be imposed as curvature of a pre-existing manifold alone. Rather, gravitational geometry should arise from deformations of the McGucken-sphere incidence relation itself. In twistor language, this means deforming the complex structure, contact structure, or infinity-twistor data that encode null incidence.

19.1 Definition 11: McGucken Gravitational Twistor Data

A McGucken gravitational twistor datum is a tuple

$$\mathfrak{G}_M = (\mathbb{PT}_M, \bar{\partial}_h, I_M, \Omega_M, \mathcal{L}_M),$$

where:

1. $\mathbb{PT}_M$ is the twistor space generated by projectivized McGucken-sphere null generators.
2. $\bar{\partial}_h = \bar{\partial} + h$ is a deformation of the complex structure of twistor space.
3. I_M is a McGucken infinity-twistor or Poisson/contact datum selecting Einstein rather than conformal-gravity degrees of freedom.
4. Ω_M is the holomorphic volume or contact measure induced by x_4-phase flux.
5. $\mathcal{L}_M$ is the worldsheet line bundle whose degree records the McGucken holomorphic-curve sector.

19.2 Definition 12: McGucken Gravitational Twistor-String Action

A minimal McGucken-informed gravitational twistor-string action has the schematic form

$$S_M = \int_\Sigma Y_I \bar{\partial}_h Z^I + S_{\text{grav}}[Z, Y; I_M, \Omega_M] + S_{x_4}[Z, Y; \rho],$$

where $Z : \Sigma \to \mathbb{PT}_M$ is a holomorphic map from the worldsheet into McGucken twistor space, Y is its conjugate worldsheet field, S_{grav} imposes Einstein-gravity vertex structure, and S_{x_4} imposes the McGucken phase-flow constraint

$$d\rho = \frac{d\alpha}{\alpha}, \qquad \frac{dx_4}{dt} = ic.$$

The role of S_{x_4} is to ensure that the worldsheet counts only those holomorphic curves compatible with coherent McGucken-sphere Huygens flow.

19.3 Theorem 28: Einstein Gravity as Deformation of McGucken-Sphere Incidence

If gravitational curvature is represented by a deformation of McGucken null-sphere incidence, then the induced twistor data are described by a deformation

$$\bar{\partial} \longmapsto \bar{\partial}_h = \bar{\partial} + h$$

together with a McGucken infinity-twistor datum I_M selecting an Einstein scale inside conformal twistor geometry.

19.3.1 Proof

In flat McGucken geometry, a spacetime point x corresponds to a twistor line CP^1_x by the incidence relation

$$\omega^A = ix^{AA'}\pi_{A'}.$$

This incidence relation is generated by the null directions of the McGucken sphere centered at x. A gravitational field changes the relation between neighboring null cones. In McGucken language, this means that the family of McGucken spheres no longer has flat incidence relations. In twistor language, changing the family of incidence lines is equivalent to deforming the complex structure of twistor space:

$$\bar{\partial} \to \bar{\partial}_h.$$

However, a complex-structure deformation by itself naturally captures conformal gravitational data. To select Einstein gravity, one must specify the additional structure that fixes a representative metric in the conformal class. Twistor constructions do this through infinity-twistor, Poisson, or contact data. Therefore the McGucken gravitational datum must include I_M, the x_4-induced structure selecting the Einstein scale.

Thus Einstein gravity is represented not merely by arbitrary deformation of twistor space, but by deformation of McGucken-sphere incidence plus the McGucken infinity-twistor datum that selects the Einstein metric. □

19.4 Theorem 29: McGucken Graviton Vertex Operators

In a McGucken gravitational twistor string, graviton vertex operators correspond to infinitesimal deformations of McGucken-sphere incidence:

$$V_h = \int_\Sigma h_I(Z)\,\bar{\partial} Z^I,$$

with the Einstein restriction implemented by I_M.

19.4.1 Proof

A vertex operator represents an infinitesimal deformation of the worldsheet action. Since the worldsheet map $Z : \Sigma \to \mathbb{PT}_M$ records holomorphic families of McGucken-sphere null generators, a deformation of the twistor complex structure changes the allowed incidence relation among those generators. Therefore an infinitesimal deformation h enters the action through

$$\bar{\partial} \to \bar{\partial} + h,$$

producing the integrated insertion

$$V_h = \int_\Sigma h_I(Z)\,\bar{\partial} Z^I.$$

This is the worldsheet image of an infinitesimal gravitational perturbation of the McGucken-sphere incidence network. The Einstein restriction is not automatic because unconstrained twistor deformations correspond to conformal gravitational data. Imposing I_M restricts the allowed deformations to those compatible with an Einstein scale. Hence V_h represents an Einstein graviton precisely when h is constrained by the McGucken infinity-twistor datum. □

19.5 Theorem 30: McGucken Rational-Curve Formula for Tree Gravity Amplitudes

At tree level, a McGucken-informed gravitational twistor string localizes n-graviton amplitudes on holomorphic maps

$$Z : \mathbb{CP}^1 \to \mathbb{PT}_M$$

whose degree is fixed by the helicity sector and whose measure is weighted by the McGucken x_4-phase-flow determinant.

Schematically,

$$\mathcal{M}^{\text{grav}}_{n,d} = \int_{\mathcal{M}_{0,n}(\mathbb{PT}_M,d)} d\mu_M \prod_{i=1}^{n} V_i^{\text{grav}},$$

where

$$d\mu_M = \det{}'_M(H)\ \det{}'_M(\widetilde{H}) \prod_e \frac{d\alpha_e}{\alpha_e}.$$

Here H and $\widetilde{H}$ are the McGucken-weighted Hodges-type matrices for the two gravitational helicity sectors, and the $d\alpha_e/\alpha_e$ factor is the x_4-phase-flow measure.

19.5.1 Proof Sketch

Known gravitational twistor formulae express tree-level gravity amplitudes in terms of rational maps into twistor space, Hodges-type determinant factors, and Grassmannian or worldsheet moduli [13–17]. In the McGucken interpretation, each insertion V_i^{grav} is a deformation of the

McGucken-sphere incidence structure associated with the i-th external graviton. The holomorphic map $Z : \mathbb{CP}^1 \to \mathbb{PT}_M$ records coherent null-generator transport through the McGucken twistor space.

The gravitational determinant factors arise because gravity couples to the geometry of incidence itself rather than to a color-ordered gauge bundle. In McGucken language, this means the amplitude weights not only paths of x_4-phase transport but also deformations of the incidence metric relating neighboring McGucken spheres. Hence the Yang-Mills Parke-Taylor/current-algebra factor is replaced by Hodges-type determinants or their higher-degree analogues.

The x_4-phase-flow coordinates satisfy

$$d\rho_e = \frac{d\alpha_e}{\alpha_e},$$

so the natural positive measure over McGucken incidence channels is

$$\prod_e \frac{d\alpha_e}{\alpha_e}.$$

Combining the gravitational determinant weight with this McGucken phase-flow measure gives the stated schematic formula. □

19.6 Theorem 31: Avoidance of Pure Conformal-Gravity Contamination

A McGucken gravitational twistor string describes Einstein gravity rather than pure conformal gravity only if the worldsheet theory includes a constraint selecting an Einstein scale:

$$I_M \neq 0, \qquad \nabla I_M = 0 \quad \text{or its twistor-equivalent holomorphic constraint.}$$

19.6.1 Proof

Twistor complex-structure deformation naturally encodes conformal geometry because null cones determine a conformal class, not a unique metric. Pure conformal gravity therefore appears when the theory permits arbitrary conformal deformations of twistor space. Einstein gravity requires more: it requires a distinguished metric representative or, equivalently, a structure fixing the conformal scale.

In twistor geometry, this role is played by infinity-twistor, Poisson, or contact data. In the McGucken framework, the same role is played by I_M, because $x_4 = ict$ fixes a physically normalized null expansion speed and therefore selects the Einstein-scale representative compatible with the invariant light-speed expansion of the McGucken sphere.

If I_M is absent, the worldsheet sums over conformal-gravity modes. If I_M is present and covariantly preserved, the allowed deformations are restricted to the Einstein sector. Therefore an Einstein gravitational twistor string requires the McGucken infinity-twistor constraint. □

19.7 Research Programme: From McGucken Spheres to Full Einstein Gravity

The resulting programme is:

$$\text{McGucken sphere} \to \text{curved null-sphere incidence} \to \text{deformed twistor space} \to \text{Einstein-scale constraint } I_M -$$

The central mathematical task is to prove that the McGucken gravitational twistor-string path integral has the Einstein-Hilbert action, rather than the conformal-gravity action, as its spacetime field-theory limit:

$$S_{\text{eff}}[\mathfrak{G}_M] = \frac{1}{16\pi G}\int_M \sqrt{-g}\,(R - 2\Lambda)\,d^4x + O(\hbar).$$

This requires four formal steps:

1. Construct $\mathbb{PT}_M$ for curved McGucken-sphere incidence.
2. Prove that I_M selects Einstein metrics inside the conformal class.
3. Derive graviton vertex operators from infinitesimal McGucken incidence deformations.
4. Show that the worldsheet correlation functions reproduce Einstein gravity amplitudes and, in the classical limit, the Einstein-Hilbert equations.

19.8 Plain-Language Explanation

Gauge theory amplitudes describe particles moving through a fixed causal background. Gravity is deeper because gravity changes the causal background itself. In McGucken language, gravity must therefore be a deformation of the McGucken spheres, not merely a field riding on top of them.

Twistor theory is already built from null rays. If gravity bends the null rays, then gravity deforms twistor space. The McGucken Principle adds the physical mechanism: the null rays are the projectivized generators of expanding McGucken spheres. Therefore a gravitational twistor string should be a string theory of deformed McGucken-sphere incidence.

The crucial point is to avoid falling back into conformal gravity. Null cones alone determine only conformal geometry. Einstein gravity requires a scale. The McGucken Principle supplies a natural scale through the invariant expansion law $dx_4/dt = ic$. This is why the McGucken gravitational twistor string must include I_M: the McGucken infinity-twistor or scale datum that selects full Einstein gravity.

20 Complete Derivation Chain

The full construction is:

$$\boxed{dx_4/dt = ic}$$

$$\Downarrow$$

$$\boxed{\Sigma_+(p) = \text{McGucken Sphere} = \text{future null cone}}$$

$$\Downarrow$$

$$\boxed{\omega^A = ix^{AA'}\pi_{A'}}$$

$$\Downarrow$$

$$\boxed{CP^1 \text{ incidence lines in } CP^3}$$

$$\Downarrow$$

$$\boxed{Z_a = (\lambda_a, x_a\lambda_a)}$$

$$\Downarrow$$

$$\boxed{Z \in M_+(k+4, n)}$$

$$\Downarrow$$

$$\boxed{C_{\alpha a} = \sum_{\gamma:\alpha\to a} \prod_{e\in\gamma} e^{\rho_e} \in G_+(k,n)}$$

$$\Downarrow$$

$$\boxed{Y = CZ}$$

$$\Downarrow$$

$$\boxed{D_{(i)} = \text{closed } x_4\text{-chain boundary measurement}}$$

$$\Downarrow$$

$$\boxed{(C, D_{(i)}) \in G_+(k,n;L)}$$

$$\Downarrow$$

$$\boxed{\mathcal{L}_{(i)} = D_{(i)} Z}$$

$$\Downarrow$$

$$\boxed{\Omega = \Phi_* \prod_i \frac{d\alpha_i}{\alpha_i}}$$

$$\Downarrow$$

$$\boxed{\text{amplituhedron canonical form}}.$$

21 Twistor-Amplituhedron Descent from the McGucken Sphere as the Foundational Atom

The preceding sections establish the individual maps from the McGucken Principle to null geometry, from null geometry to twistor incidence, from twistor incidence to momentum-twistor kinematics, from positive x_4-flow networks to the positive Grassmannian, and from the positive Grassmannian to the amplituhedron map. This section records the complete descent in one theorem and makes explicit the central claim: twistors and the amplituhedron are not external mathematical structures imposed upon the McGucken Principle, but successive projective and positive-geometric consequences of the McGucken sphere understood as the foundational atom of spacetime.

More strongly, the McGucken sphere is not only the foundational geometric object from which twistors and the amplituhedron are constructed; it is the foundational atom of spacetime itself. Twistor space is obtained by projectivizing its null generators, while the amplituhedron is obtained by organizing many such McGucken-sphere incidence data into positive Grassmannian networks and mapping them by $Y = CZ$.

21.1 Conceptual Hierarchy

The hierarchy is:

McGucken sphere as foundational atom → null generators → twistors → momentum twistors → positive exter

Equivalently:

$$\boxed{\text{McGucken sphere as foundational atom} \to CP^1 \text{ incidence lines} \to CP^3 \text{ twistor space} \to \text{momentum-twistor p}}$$

The distinction between twistors and the amplituhedron is important. Twistor space descends directly from the projectivized null-generator geometry of a single McGucken sphere, the foundational atom of spacetime. The amplituhedron descends one structural level later: it is the positive-geometric image of many ordered McGucken-sphere incidence data after twistorization, positivity, Grassmannian boundary measurement, and the Huygens superposition map.

Thus the most precise statement is:

> Twistor space is the projectivized null-generator geometry of the McGucken sphere, the foundational atom of spacetime, while the amplituhedron is the positive-geometric image of ordered McGucken-sphere intersection networks in twistor space.

21.2 Standard Objects and McGucken Interpretation

Standard amplituhedron object	**McGucken interpretation**
Null external momentum	Relation between neighboring McGucken-sphere centers
Momentum twistor Z_a	Twistorized McGucken null-incidence datum
Ordered external data $Z_1, \ldots, Z_n$	Ordered family of McGucken-sphere null generators
Positive Grassmannian matrix C	Boundary-measurement matrix of positive x_4-directed Huygens flow
BCFW bridge	Elementary McGucken-sphere intersection/transfer channel
Positroid cell	Reduced McGucken intersection network
$Y = CZ$	Huygens superposition of twistorized sphere data
Canonical $d\log$ form	Additive x_4-phase/flux measure pushed forward to positive geometry
Amplituhedron	Positive-geometric image of all allowed ordered McGucken intersection data
Foundational spacetime atom	The McGucken sphere as the primitive null-incidence unit generating the above

21.3 Theorem 25: Twistor-Amplituhedron Descent from the McGucken Sphere as the Foundational Atom of Spacetime

Let Σ_p denote the McGucken sphere centered at event p, defined by the null condition induced by $dx_4/dt = ic$. Treat Σ_p as the foundational atom of spacetime: the primitive causal-incidence unit associated with the event p. Then:

1. The projective null-generator space of Σ_p is the twistor incidence line $CP^1_p \subset CP^3$.
2. An ordered planar family of McGucken spheres $\{\Sigma_{p_a}\}_{a=1}^n$ determines ordered momentum-twistor data Z_a.
3. If the corresponding x_4-phase ordering is positive, the data define $Z \in M_+(k+4, n)$.
4. Reduced positive McGucken intersection networks define boundary-measurement matrices $C \in G_+(k, n)$.
5. The Huygens superposition map sends this data to
$$Y = CZ.$$

Therefore the tree amplituhedron is the positive-geometric image of ordered McGucken-sphere intersection data built from the foundational spacetime atom Σ_p.

21.3.1 Proof

By the McGucken Principle,

$$\frac{dx_4}{dt} = ic,$$

integration gives $x_4 = ict$, up to an additive constant. Substitution into the four-dimensional Euclideanized interval gives

$$ds^2 = dx^2 + dy^2 + dz^2 + dx_4^2 = dx^2 + dy^2 + dz^2 - c^2dt^2.$$

Therefore $ds^2 = 0$ gives the null-sphere condition

$$dx^2 + dy^2 + dz^2 = c^2dt^2.$$

Thus each spacetime event p generates an expanding null McGucken sphere Σ_p. This is why Σ_p is the foundational atom of spacetime in the present construction: it is the event-level generator of null incidence. Its null generators are labeled by projective spinors $\pi_{A'}$. The corresponding twistor incidence relation is

$$\omega^A = ip^{AA'}\pi_{A'}.$$

For fixed p, varying $\pi_{A'}$ over projective spinor space gives a projective line $CP_p^1 \subset CP^3$. Hence the projectivized generator space of Σ_p is exactly the twistor incidence line associated with p.

For a planar scattering process, introduce region momenta x_a such that adjacent differences are null:

$$p_a = x_a - x_{a-1}, \qquad p_a^2 = 0.$$

In the McGucken construction, each region point x_a is the center of a McGucken sphere, and each null separation $x_a - x_{a-1}$ is the shared null incidence relation between neighboring spheres. The twistorized form of this planar null-polygon data is the momentum-twistor datum Z_a.

If the external ordering follows the monotone x_4-phase orientation, then the ordered maximal minors of the external twistor matrix are positive. Consequently the ordered external data define

$$Z \in M_+(k+4, n).$$

Now consider a reduced McGucken intersection network. Each internal channel carries a positive Huygens-flow weight $\alpha_e > 0$. Its boundary measurement matrix is

$$C_{\alpha a} = \sum_{\gamma:\alpha\to a} \prod_{e\in\gamma} \alpha_e.$$

Because every path contribution is a product of positive weights, and because reduced planar networks give positroid coordinate charts, C lies in the positive Grassmannian:

$$C \in G_+(k, n).$$

Finally, Huygens superposition states that the output Y_α is the positive linear combination of external twistorized McGucken-sphere data with coefficients $C_{\alpha a}$:

$$Y_\alpha^I = \sum_{a=1}^{n} C_{\alpha a} Z_a^I.$$

This is precisely the amplituhedron map:

$$Y = CZ.$$

Therefore the tree amplituhedron is the positive image of ordered McGucken-sphere intersection data. Since the McGucken sphere is the primitive null-incidence unit associated with every event, the amplituhedron is ultimately a positive-geometric image of networks of foundational spacetime atoms. □

21.4 Corollary 5: Direct and Indirect Descent

Twistor space descends directly from the McGucken sphere as the foundational atom of spacetime, whereas the amplituhedron descends indirectly through ordered planar twistor data, positivity, the positive Grassmannian, and Huygens superposition.

21.4.1 Proof

The direct descent is the incidence construction:

$$\Sigma_p \to \{\text{null generators of } \Sigma_p\}/\mathbb{C}^* \to CP^1_p \subset CP^3.$$

The indirect descent is the many-particle positive construction:

$$\{\Sigma_{p_a}\}_{a=1}^n \to \{Z_a\}_{a=1}^n \to Z \in M_+(k+4, n) \to C \in G_+(k, n) \to Y = CZ.$$

Thus twistor space is the projective null-generator geometry of a McGucken sphere, while the amplituhedron is the positive-geometric image of ordered networks of such sphere data. In this precise sense, both twistors and the amplituhedron descend from the McGucken sphere as the foundational atom of spacetime. □

21.5 Plain-Language Explanation

The McGucken sphere gives the light cone, and therefore acts as the foundational atom of spacetime. Twistor theory is what results when the light cone is not treated primarily as a set of spacetime points, but as a family of null rays. A twistor is a projective label for one of those null rays.

The amplituhedron arises after this construction is repeated for many scattering particles. Each external particle contributes null incidence data. Those data are organized into momentum twistors. Positivity then selects the physically allowed orientation and ordering. Positive Huygens-flow networks encode how the null-sphere data may be recombined. The map $Y = CZ$ sends those positive combinations into a geometric region: the amplituhedron.

Thus the McGucken sphere is the primitive geometric object: the foundational atom of spacetime. Its null generators, when projectivized, give twistor incidence. Ordered families of such incidence data give momentum twistors. Positive Huygens-flow networks among these twistors give the positive Grassmannian. The amplituhedron is then the image of these positive McGucken networks under $Y = CZ$. Therefore twistors and the amplituhedron are not independent mathematical miracles appended to quantum field theory; they are successive projective and positive-geometric consequences of the expanding fourth-dimensional McGucken sphere.

22 Completed Results

Target	Result
McGucken sphere	Derived from $dx_4/dt = ic$ as the future null cone and identified as the foundatio
Twistor space	Derived from McGucken incidence $\omega^A = ix^{AA'}\pi_{A'}$.
Momentum twistors	Derived as planar null-polygon McGucken incidence data.
External positivity	Classified as $M_+(k+4, n)/GL^+(k+4)$.
Witten curve degree	Matched exactly by $d = k_\text{A} + 1 + \ell$.
Positive Grassmannian	Derived from boundary measurements of positive x_4-flux networks.
BCFW cells	Derived from McGucken BCFW bridges.
Positroid cells	Derived from reduced McGucken networks.
Amplituhedron map	Derived as Huygens superposition $Y = CZ$.
Canonical forms	Derived from $d\rho = d\alpha/\alpha$ and pushforward.
Boundaries	Derived from degeneration of x_4-flux channels.
Locality	Derived from null McGucken-sphere separations.
Unitarity	Derived from opened closed x_4-chains.
Loop data	Derived from $D_{(i)}$ boundary measurements.
Yangian invariance	Derived from ordinary plus dual conformal McGucken symmetry.
Foundational-atom thesis	The McGucken sphere is the primitive null-incidence unit from which twistor sp
Operator-algebraic microcausality	Derived as graded commutativity of local algebras whose McGucken causal com
Einstein gravitational twistor string	Formulated as a deformation of McGucken-sphere incidence with an infinity-twis

23 Conclusion

The McGucken Principle $dx_4/dt = ic$ gives a single geometric origin for the chain of structures leading to the amplituhedron. That origin is the McGucken sphere, the foundational atom of spacetime. The fourth-dimensional expansion produces McGucken spheres, whose projectivized null incidence produces Penrose twistor space. Planar scattering organizes these null directions into region-momentum polygons, producing momentum twistors. Ordered x_4-phase gives positive external data. Directed x_4-flux networks give positive Grassmannian matrices by boundary measurement. Huygens superposition gives $Y = CZ$. Additive x_4-flux coordinates produce canonical $d\log$ forms. Degenerations of the networks give residues, factorization, locality, unitarity, and positroid stratification. Closed x_4-chains give loop matrices $D_{(i)}$, yielding the full loop amplituhedron.

Thus the amplituhedron is not merely compatible with the McGucken Principle. In the present construction, Penrose twistor space derives from the projectivized null-generator geometry of the McGucken sphere, and the amplituhedron derives from the ordered positive geometry of McGucken-sphere intersection networks. Equivalently, the amplituhedron is the positive-geometric image of x_4-generated Huygens phase-flow after twistorization, positivity, and Grassmannian boundary measurement.

The McGucken sphere is therefore the primitive geometric object and the foundational atom of spacetime. It is not only the foundational geometric object from which twistors and the amplituhedron are constructed; it is the elementary spacetime atom whose null incidence generates the geometry on which those structures rest. Its null generators, when projectivized, give twistor incidence. Ordered families of such incidence data give momentum twistors. Positive Huygens-flow networks among these twistors give the positive Grassmannian. The amplituhedron is then the image of these positive McGucken networks under $Y = CZ$. Therefore twistors and the amplituhedron are successive projective and positive-geometric consequences of the expanding fourth-dimensional McGucken sphere.

The remaining structural frontier now has a precise formulation. Operator-algebraic microcausality becomes the statement that local algebras commute whenever their McGucken causal completions are disjoint. Full Einstein gravity becomes the problem of constructing a gravitational twistor string whose worldsheet fields deform McGucken-sphere incidence while an x_4-induced infinity-twistor datum selects the Einstein sector rather than conformal gravity. These two extensions move the McGucken programme from scattering geometry toward a full algebraic and gravitational foundation.

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